INF01056
AULA 09/10
COMBINATÓRIA

PROF. JOÃO COMBA

BASEADO NO LIVRO PROGRAMMING CHALLENGES

TÉCNICAS DE CONTAGEM

- *Product Rule* — The *product rule* states that if there are $|A|$ possibilities from set A and $|B|$ possibilities from set B, then there are $|A| \times |B|$ ways to combine one from A and one from B. For example, suppose you own 5 shirts and 4 pants. Then there are $5 \times 4 = 20$ different ways you can get dressed tomorrow.

- *Sum Rule* — The *sum rule* states that if there are $|A|$ possibilities from set A and $|B|$ possibilities from set B, then there are $|A| + |B|$ ways for either A *or* B to occur – assuming the elements of A and B are distinct. For example, given that you own 5 shirts and 4 pants and the laundry ruined one of them, there are 9 possible ruined items.[1]

TÉCNICAS DE CONTAGEM

- *Inclusion-Exclusion Formula* — The sum rule is a special case of a more general formula when the two sets can overlap, namely,

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

  For example, let A represent the set of colors of my shirts and B the colors of my pants. Via inclusion-exclusion, I can calculate the total number of colors given the number of color-matched garments or vice versa. The reason this works is that summing the sets double counts certain possibilities, namely, those occurring in both sets. The inclusion-exclusion formula generalizes to three sets and beyond in a natural way:

  $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

TÉCNICAS DE CONTAGEM

- *Permutations* — A *permutation* is an arrangement of n items, where every item appears exactly once. There are $n! = \prod_{i=1}^{n} i$ different permutations. The $3! = 6$ permutations of three items are 123, 132, 213, 231, 312, and 321. For $n = 10$, $n! = 3,628,800$, so we start to approach the limits of exhaustive search.
- *Subsets* — A *subset* is a selection of elements from n possible items. There are 2^n distinct subsets of n things. Thus there are $2^3 = 8$ subsets of three items, namely, 1, 2, 3, 12, 13, 23, 123, and the empty set: never forget the empty set. For $n = 20$, $2^n = 1,048,576$, so we start to approach the limits of exhaustive search.
- *Strings* — A *string* is a sequence of items which are drawn *with repetition.* There are m^n distinct sequences of n items drawn from m items. The 27 length-3 strings on 123 are 111, 112, 113, 121, 122, 123, 131, 132, 133, 211, 212, 213, 221, 222, 223, 231, 232, 233, 311, 312, 313, 321, 322, 323, 331, 332, and 333. The number of binary strings of length n is identical to the number of subsets of n items (why?), and the number of possibilities increases even more rapidly with larger m.

RELAÇÕES DE RECORRÊNCIA

$$a_n = a_{n-1} + 1, a_1 = 1 \longrightarrow a_n = n$$

$$a_n = 2a_{n-1}, a_1 = 2 \longrightarrow a_n = 2^n$$

$$a_n = na_{n-1}, a_1 = 1 \longrightarrow a_n = n!$$

COEFICIENTES BINOMIAIS

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1} = \frac{n!}{k!(n-k)!} \quad \text{if } n \geq k \geq 0$$

- *Committees* — How many ways are there to form a k-member committee from n people? By definition, $\binom{n}{k}$ is the answer.
- *Paths Across a Grid* — How many ways are there to travel from the upper-left corner of an $n \times m$ grid to the lower-right corner by walking only down and to the right? Every path must consist of $n + m$ steps, n downward and m to the right. Every path with a different set of downward moves is distinct, so there are $\binom{n+m}{n}$ such sets/paths.
- *Coefficients of* $(a+b)^n$ — Observe that

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

What is the coefficient of the term $a^k b^{n-k}$? Clearly $\binom{n}{k}$, because it counts the number of ways we can choose the k a-terms out of n possibilities.

Triângulo de Pascal

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$(n+1)$st row of the table gives the values $\binom{n}{i}$ for $0 \le i \le n$.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Exemplo

How many different ordered triples (a,b,c) of non-negative integers are there such a+b+c=50 ?

EXEMPLO

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a+b=n n+1 solutions

(0,n)(1,n-1),(2,n-2),...,(n,0)

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```
 c=0, a+b=50
 c=1, a+b=49
 c=2, a+b=48   1+2+3+...+51 = (51*52)/2 = C(52,2)
     ...
c=37, a+b=33
```

Exemplo

How many different ordered triples (a,b,c) of non-negative integers are there such $a+b+c=11$?

●●● + ●●●●●● + ●● = ●●●●●●●●●●●

$C(13,2)$

How many different ordered triples (a,b,c) of non-negative integers are there such $a+b+c=50$?

$C(52,2)$

Exemplo

How many different ways we can place b indistinguishable into u distinguishable urns ?

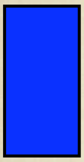
$C(b+u-1,b) = C(b+u-1, u-1)$

$B=50$ $U=3$

$C(52, 2)$

Outras Seqüências

Define a domino to be a 1x2 rectangle. In how many ways can an nx2 rectanle be tiled by dominos ?



$T_1 = 1$

Outras Seqüências

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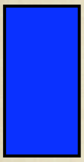

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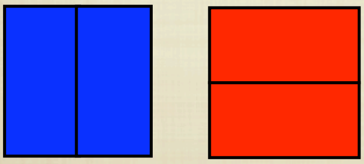
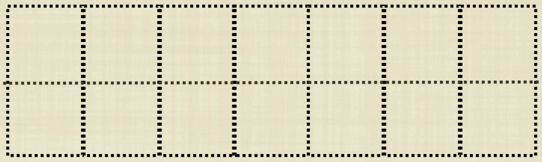


$T_2 = 2$

Outras Seqüências

Define a domino to be a 1x2 rectangle. In how many ways can an nx2 rectanle be tiled by dominos ?

T_7 = ?



Outras Seqüências

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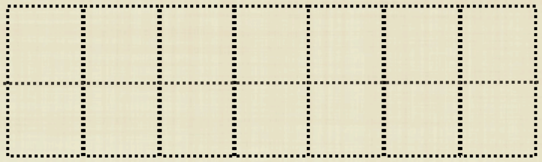
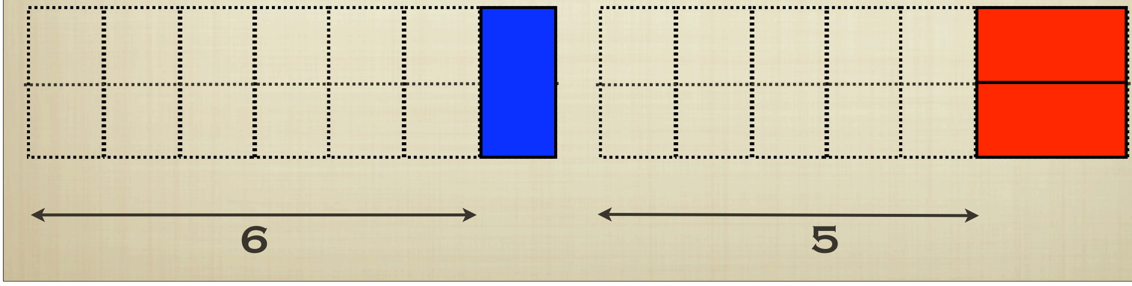


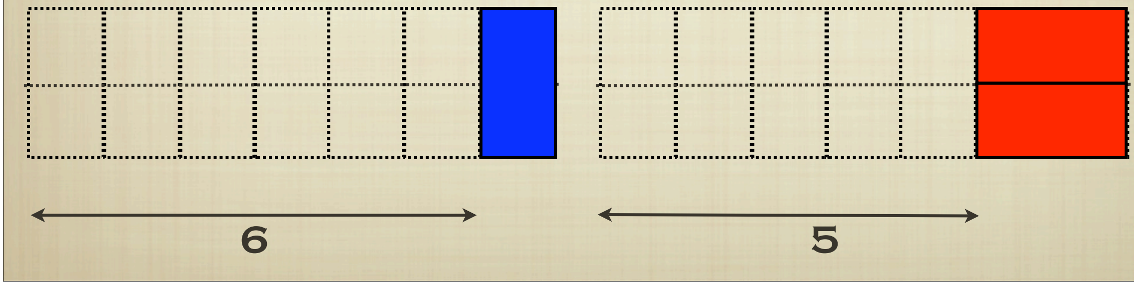


Outras Seqüências

Define a domino to be a 1x2 rectangle. In how many ways can an nx2 rectanle be tiled by dominos ?

$T_7 = T_6 + T_5$

$T_{N+1} = T_N + T_{N-1}$
$T_1 = 1,\ T_2 = 2$



Outras Seqüências

3 numbers to make a proportion

$$\frac{a}{b} = \frac{b}{c}$$

2 numbers to make a proportion

$$\frac{a}{b} = \frac{b}{a+b}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$\left(\frac{b}{a}\right)^2 = \left(\frac{b}{a}\right) + 1$$

$$GoldenRatio\ \varphi = \frac{1 + \sqrt{5}}{2}$$

$$x^2 - x - 1 = 0$$

OUTRAS SEQÜÊNCIAS

$$\begin{aligned} x &= \lim_{n\to\infty} \frac{F(n+1)}{F(n)} \\ &= \lim_{n\to\infty} \frac{F(n)+F(n-1)}{F(n)} \\ &= \lim_{n\to\infty} \left(\frac{F(n)}{F(n)} + \frac{F(n-1)}{F(n)}\right) \\ &= 1 + \lim_{n\to\infty} \frac{F(n-1)}{F(n)} \\ &= 1 + \frac{1}{\lim\limits_{n\to\infty} \dfrac{F(n)}{F(n-1)}} \\ &= 1 + \frac{1}{x} \end{aligned}$$

$$\frac{x}{1} = \frac{1}{x-1}$$

JOHANNES KEPLER

OUTRAS SEQÜÊNCIAS

- *Fibonacci numbers* — Defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ and the initial values $F_0 = 0$ and $F_1 = 1$, they emerge repeatedly because this is perhaps the simplest interesting recurrence relation. The first several values are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … The Fibonacci numbers lend themselves to an amazing variety of mathematical identities, and are just fun to play with. They have the following hard-to-guess but simple-to-derive closed form:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

This closed form has certain important implications. Since $(1-\sqrt{5})/2$ is between 0 and 1, raising it to any power leaves a number in this range. Thus the first term, ϕ^n where $\phi = (1+\sqrt{5})/2$ is the driving quantity, and can be used to estimate F_n to within plus or minus 1.

Outras Seqüências

- How many ways are there to build a balanced formula from n sets of left and right parenthesis ?
  - for n = 3 ?

Outras Seqüências

- How many ways are there to build a balanced formula from n sets of left and right parenthesis ?
  - for n = 3 ? there are 5 different ways:
    - ((()))
    - ()(())
    - (())()
    - (()())
    - ()()()

OUTRAS SEQÜÊNCIAS

- HOW MANY WAYS ARE THERE TO BUILD A BALANCED FORMULA FROM N SETS OF LEFT AND RIGHT PARENTHESIS ?
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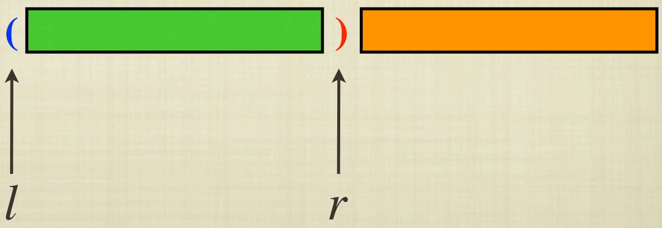

LEFMOST

OUTRAS SEQÜÊNCIAS

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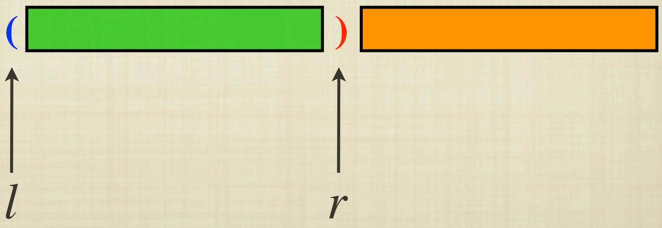

BALANCED BALANCED

Outras Seqüências

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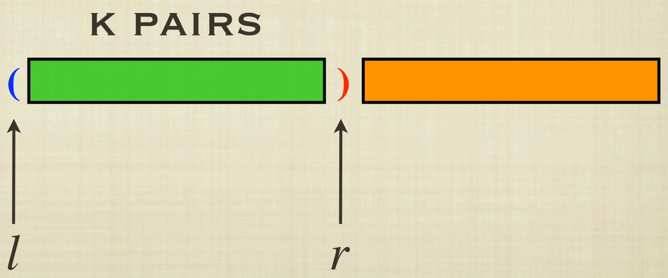

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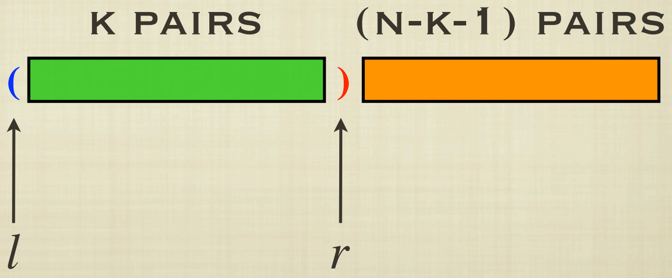

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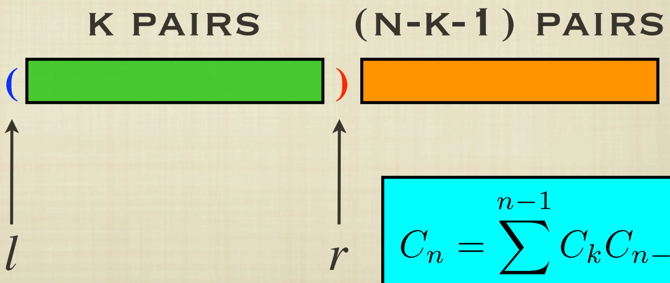
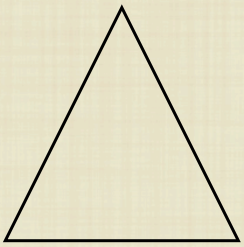


$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

OUTRAS SEQÜÊNCIAS

- HOW MANY WAYS TO TRIANGULATE A CONVEX POLYGON ?



$T_3 = 1$

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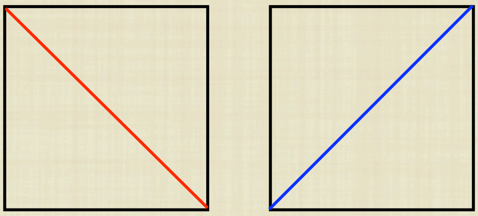
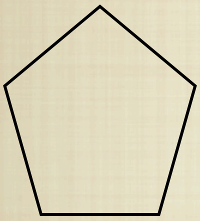


$T_4 = 2$

Outras Seqüências

- How many ways to triangulate a convex polygon ?



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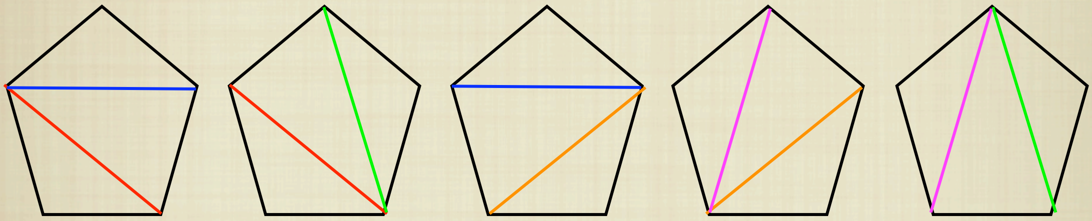
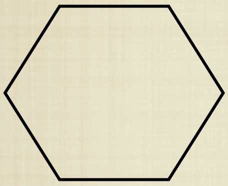


$T_4 = 5$

Outras Seqüências

- How many ways to triangulate a convex polygon ?



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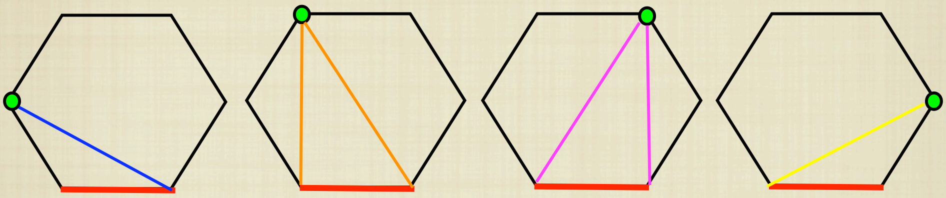

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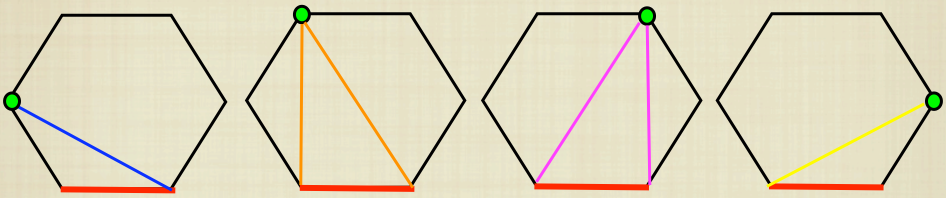


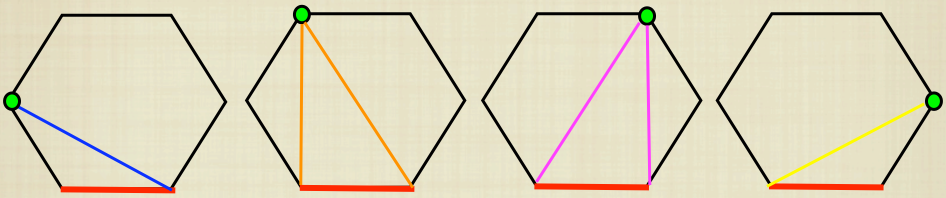
$T_6 = T_3 * T_5 + T_3 * T_4 + T_4 * T_3 + T_5 * T_3$

Outras Seqüências

- How many ways to triangulate a convex polygon ?

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

$T_4 = C_2$
$T_5 = C_3$
$T_6 = C_4$
$C_0 = 1$



$T_6 = T_3 * T_5 + T_3 * T_4 + T_4 * T_3 + T_5 * T_3$

$T_6 = 1*5 + 1*2 + 2*1 + 5*1 = 14$

OUTRAS SEQÜÊNCIAS

- *Catalan Numbers* — The recurrence and associated closed form

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} = \frac{1}{n+1}\binom{2n}{n}$$

defines the *Catalan numbers*, which occur in a surprising number of problems in combinatorics. The first several terms are $2, 5, 14, 42, 132, 429, 1430, \ldots$ when $C_0 = 1$.

OUTRAS SEQÜÊNCIAS

- *Eulerian Numbers* — The *Eulerian* numbers $\left\langle {n \atop k} \right\rangle$ count the number of permutations of length n with exactly k ascending sequences or *runs*. A recurrence can be formulated by considering each permutation p of $1, ..., n-1$. There are n places to insert element n, and each either splits an existing run of p or occurs immediately after the last element of an existing run, thus preserving the run count. Thus $\left\langle {n \atop k} \right\rangle = k\left\langle {n-1 \atop k} \right\rangle + (n-k+1)\left\langle {n-1 \atop k-1} \right\rangle$. Can you construct the eleven permutations of length four with exactly two runs?

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1324-1423-2314-2413-3412

1243-1342-2341-2134-3124-4123

OUTRAS SEQÜÊNCIAS

- *Stirling Numbers* — There are two different types of Stirling numbers. The first, $\left[{n \atop k} \right]$, counts the number of permutations on n elements with exactly k cycles. To formulate the recurrence, observe the nth element either forms a singleton cycle or it doesn't. If it does, there are $\left[{n-1 \atop k-1} \right]$ ways to arrange the rest of the elements to form $k-1$ cycles. If not, the nth element can be inserted in every possible position of every cycle of the $\left[{n-1 \atop k} \right]$ ways to make k cycles out of $n-1$ elements. Thus

$$\left[{n \atop k} \right] = \left[{n-1 \atop k-1} \right] + (n-1)\left[{n-1 \atop k} \right]$$

There are 11 permutations of four elements with exactly two cycles.

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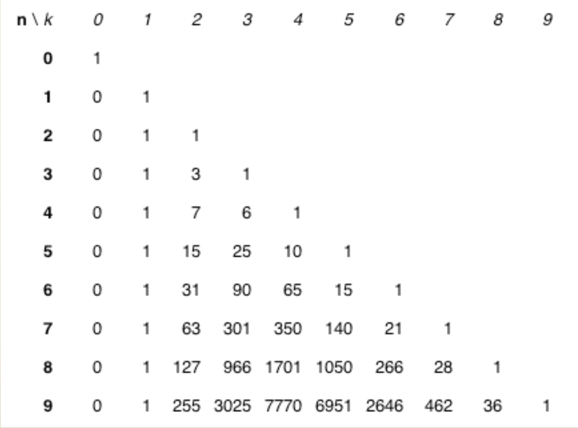
There are 11 permutations of four elements with exactly two cycles.

[1,2,3][4] [1,2,4][3] [1,3,4][2] [2,3,4][1]
[1,3,2][4] [1,4,2][3] [1,4,3][2] [2,4,3][1]
[1,2][3,4] [1,3][2,4] [1,4][2,3]

OUTRAS SEQÜÊNCIAS

- *Set Partitions* — The second kind of Stirling number $\left\{{n \atop k}\right\}$ counts the number of ways to partition n items into k sets. For example, there are seven ways to partition four items into exactly two subsets: (1)(234), (12)(34), (13)(24), (14)(23), (123)(4), (124)(3) and (134)(2). The nth item can be inserted into any of the k subsets of an $n-1$-part partition or it forms a singleton set. Thus by a similar argument to that of the other Stirling numbers they are defined by the recurrence $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k}\right\}$. The special case of $\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, since any proper subset of the elements 2 to n can be unioned with (1) to define the set partition. The second part of the partition consists of exactly the elements not in this first part.



n \ *k*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

OUTRAS SEQÜÊNCIAS

- *Integer Partitions* — An integer partition of n is an unordered set of positive integers which add up to n. For example, there are seven partitions of 5, namely, (5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), and (1, 1, 1, 1, 1). The easiest way to count them is to define a function $f(n, k)$ giving the number of integer partitions of n with largest part at most k. In any acceptable partition the largest part either does or does not reach with limit, so $f(n, k) = f(n-k, k) + f(n, k-1)$. The basis cases are $f(1, 1) = 1$ and $f(n, k) = 0$ whenever $k > n$.

RECURSÃO E INDUÇÃO

For example, consider the following recurrence relation:

$$T_n = 2T_{n-1} + 1, T_0 = 0$$

Building a table of values yields the following:

n	0	1	2	3	4	5	6	7
T_n	0	1	3	7	15	31	63	127

Can you guess what the solution is? You should notice that things look like they are doubling each time, no surprise considering the formula. But it is not quite 2^n. By playing around with variations of this function, you should be able to stumble on the conjecture that $T_n = 2^n - 1$. To finish the job, we must prove this conjecture, using the three steps of induction:

1. Show that the basis is true: $T_0 = 2^0 - 1 = 0$.
2. Now assume it is true for T_{n-1}.
3. Use this assumption to complete the argument:

$$T_n = 2T_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$$