BASICS OF ALGORITHM ANALYSIS

Marcus Ritt

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Outline

1. Intro
2. Stable matchings

Our motto is:

1. An algorithmic idea.
2. A proof that the idea works.
3. A definition of the data structures and operations needed to make it work.
4. An analysis of its complexity based on the defines operations.
5. An efficient implementation of the data structures and a refined analysis.

- 1962; Nobel in economics in 2012 for Roth & Shapley "for the theory of stable allocations and the practice of market design".
- The problem: n women, n men, each w/ a priority list of the members of the opposite sex. How to match them best? What's best? Simpler: find a *stable* matching.
- Let the women be $w_1, \ldots, w_n$ and the men $m_1, \ldots, m_n$ and let $r(m_i, w_j) \in [n]$ be the rank of woman w_j on m_i's list.
- When is a matching *unstable*?

Repeated stabilization

How to find a stable matching?

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while not stable: switch
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Main idea: Each man goes over his candidate list in order, and proposes to the next candidate, and then stops. The engaged pairs marry.

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while there's a free man m with a candidate do
  m proposes to its (current best) candidate w
  case w is free: (m,w) get engaged
  case w is engaged with m':
    if r(w,m)<r(w,m')
      m' is set free
      (m,w) get engaged
    else
      m remains free
      (m',w) remain engaged
end
return the engaged pairs
```

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Complexity

Observations:

- Each proposal (m,w) happens at most once.
- There are at most n^2 pairs.-
- Thus: the algorithm terminates in $O(n^2)$ iterations.

Costs of an iteration?

- There exist several stable matchings
- A pair (m,w) is *valid* if there's some stable matching containing (m,w)
- For each man m we can define best(m) as the best ranked woman such that (m,best(m)) is valid
- Similarly we can define worst(m), and the same for the women
- Fact: Our algorithm finds always (m,best(m)) for all men, and (worst(w),w) for all women
- If the women propose, this is inverted