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BASICS OF ALGORITHM ANALYSIS

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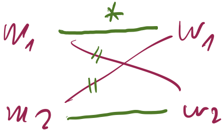
Outline

1. Stable matchings
2. Representative problems and typical complexities
3. Other problems

- The problem: n women, n men, each w/ a priority list of the members of the opposite sex. How to match them best? What's best? Simpler: find a *stable* matching.
- Let the women be $w_1,\ldots,w_n$ and the men $m_1,\ldots,m_n$ and let $r(m_i,w_j)\in[n]$ be the rank of woman w_j on m_i's list.
- When is a matching *unstable*?



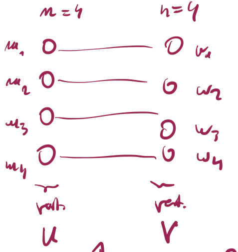
How to find a stable matching?

`while not stable: switch`

$\leadsto$ ñ precisa convergir

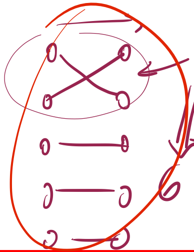
Ideias?

$n=4$ $\quad$ $n=4$



Teste: estável?

$O(n^2)$

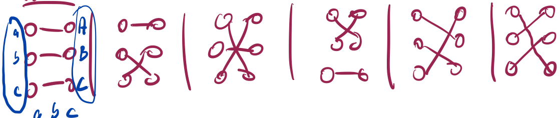


instável

Força bruta:

$n=2$

$n=3$



ABC $\quad$ ACB $\quad$ CBA $\quad$ BAC $\quad$ BCA $\quad$ CAB

$\{A, B, C\}$

Emp. perfeito
$\iff$ Emp. grafos bipartidos
$\iff$ Permutações

Corr.: trivial

Complexidade: $O(n^2 \, n!) = O^*(n!)$

EXPONENCIAL em n

Stirling: $\approx (n/e)^n \approx n^{(n)}$

Main idea: Each man goes over his candidate list in order, and proposes to the next candidate, and then stops. The engaged pairs marry.

```
while there's a free man m with a candidate do
  m proposes to its (current best) candidate w
  case w is free: (m,w) get engaged
  case w is engaged with m':
    if r(w,m)<r(w,m')
      m' is set free
      (m,w) get engaged
    else
      m remains free
      (m',w) remain engaged
end
return the engaged pairs
```

$O(x)$

$\#\text{it.} \leq n^2$

$O(n^2 \cdot x)$

$\Rightarrow O(n^2)$

$\cancel{O(n^3)}$

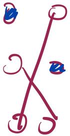
$\Omega(n^2)$

$\Rightarrow$ ex.

Corretude:

1) O resultado é um emp. perfeito ✓

2) O emp. perf. é estável.



ad 1) Obs.

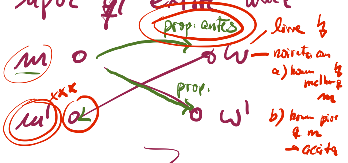
i) No final nenhuma mulher é livre, porque ela recebeu pelo menos uma proposta, e mulheres sempre ficam em noivado.

ii) No final nenhum homem é livre.

i, ii) $\Rightarrow$ 1).

ad 2) Prova por contradição: Supõe q/ existe uma instabilidade

$$\begin{bmatrix} r(m,w) < r(m,w') \\ r(w,m) < r(w,m') \end{bmatrix}$$



Observations:

- Each proposal (m,w) happens at most once.
- There are at most n^2 pairs.
- Thus: the algorithm terminates in $O(n^2)$ iterations.

Cost of an iteration?

↯ $\Rightarrow$ 2)

Obs O ranque (do noivo) de mulheres somente melhora. (diminui)

Por iteração

$l_i \leq n$

Homem $i \in [n]$: l_i Candidata atual

Emparelhamento: $m[i]$: mulher emp. ou Θ.

$w[i]$: homem emp. ou Θ.

$(m_5, w_7) \iff m[5] := 7$
$w[7] := 5$

(m_5, w_7)
$m[5] := 0$
$w[7] := 0$

Lista de homens livres (ini: $O(n)$;
extrair um homem livre $O(1)$).

Invariante: Proposição V/F

↳ Não existe instabilidade no emp. parcial (ñ perfeito). ✓

1ª: V
2ª: V?

⇓

+ Obs 1 (perfeito)

⇒ Corr.

Correctness

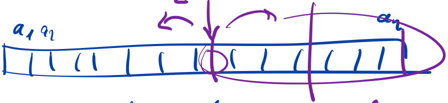
- There exist several stable matchings
- A pair (m,w) is *valid* if there's some stable matching containing (m,w)
- For each man m we can define best(m) as the best ranked woman such that (m,best(m)) is valid
- Similarly we can define worst(m), and the same for the women
- Fact: Our algorithm finds always (m,best(m)) for all men, and (worst(w),w) for all women
- If the women propose, this is inverted

Binary search

Busca binária

$<$ $=$ $>$

$a_1\ a_2$ a_n



$a_1 \leq a_2 \leq \dots \leq a_n$

$v \in a$? S

$n \to n/2 \to n/4$

$n/2^d \leq 1$

$d = 0, 1, \dots$

$\iff \lceil \log_2 n \leq d \rceil$

$\lceil \log_2 n \rceil$

Complexidade de tempo:

1) Sublinear $o(n)$ — $\log n$ — tem entr.

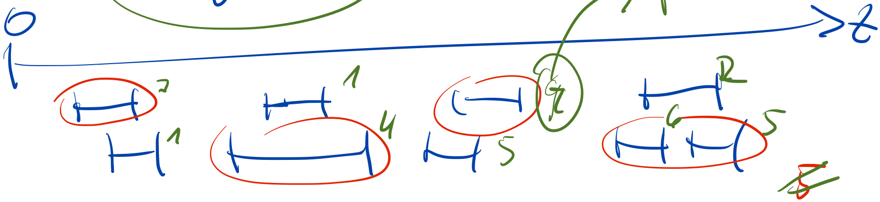
2) Linear $\Theta(n)$ — $n \log n$

3) Polinomiais $O(n^k)$ $= n^{O(1)}$ $\cdot \log n$

4) Exponenciais ex. $\Theta(2^n)$ $O(n^n)$

~~Polynomial~~

→ peso PD



Sel. maior # intervalos sem sobreposição.

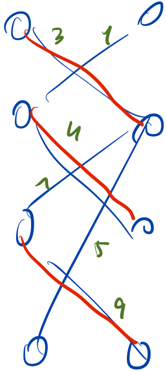
↳ Guloso

$3+4+9+6+5=$
$\underline{27}$

Weighted interval scheduling

Bipartite matching

Grafos gerais



Maior # de arestas

16

maior peso total de arestas

Polynomial

Independent set