EXACT APPROACHES TO ZONING PROBLEMS

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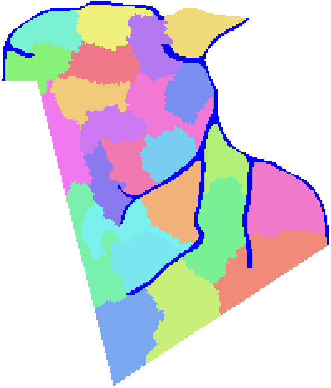
I Workshop de Problemas de Parcelamento Territorial — 30 de agosto 2018

Outline

1. Overview
2. Models and algorithms
3. Summary and outlook

The problem

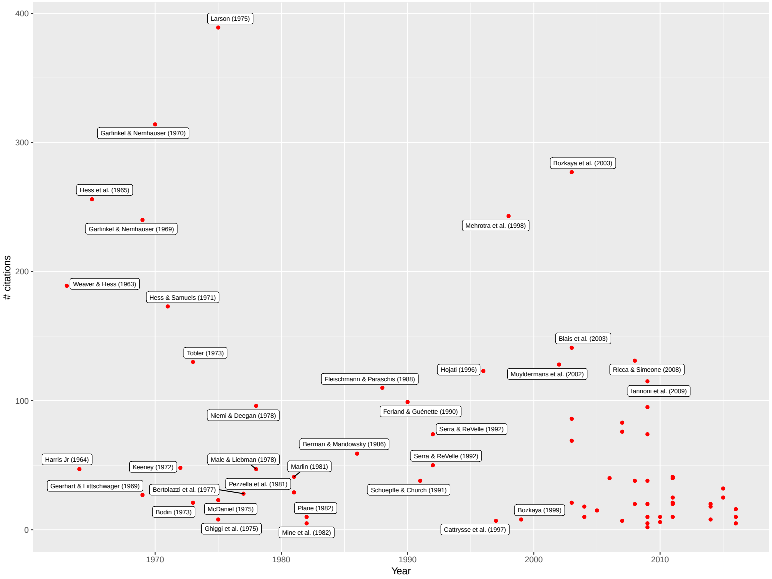
- Partition a geographical region into smaller zones
  - Usually made up of *basic units*
- Known as *districting*, *territory design*, *zoning*



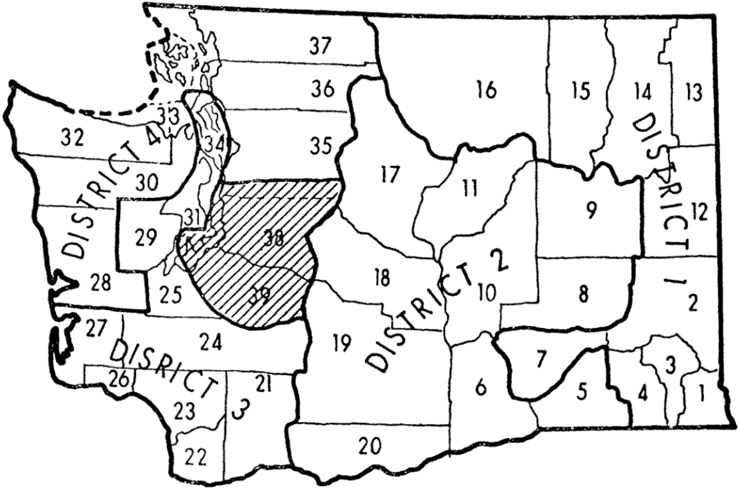
(Gliesch et al. 2017)

Applications

- Political or school districting
- Services
  - Commerce
  - Health-care, home care, emergency services
  - Fire stations, police stations
  - Electrical power distribution
  - Road maintenance
  - Gritting sand salt spreading (in cold countries)


Larson (1975)
Garfinkel & Nemhauser (1970)
Bozkaya et al. (2003)
Hess et al. (1965)
Garfinkel & Nemhauser (1969)
Mehrotra et al. (1998)
Weaver & Hess (1963)
Hess & Samuels (1971)
Blais et al. (2003)
Tobler (1973)
Hojati (1996)
Muyldermans et al. (2002)
Ricca & Simeone (2008)
Fleischmann & Paraschis (1988)
Iannoni et al. (2009)
Niemi & Deegan (1978)
Ferland & Guénette (1990)
Serra & ReVelle (1992)
Berman & Mandowsky (1986)
Harris Jr (1964)
Keeney (1972)
Male & Liebman (1978)
Marlin (1981)
Serra & ReVelle (1992)
Gearhart & Liittschwager (1969)
Bertolazzi et al. (1977)
Pezzella et al. (1981)
Schoepfle & Church (1991)
Bodin (1973)
McDaniel (1975)
Plane (1982)
Bozkaya (1999)
Ghiggi et al. (1975)
Mine et al. (1982)
Cattrysse et al. (1997)
citations
0
100
200
300
400
1970
1980
1990
2000
2010
Year

37 counties grouped into 4 political districts, Washington, USA.

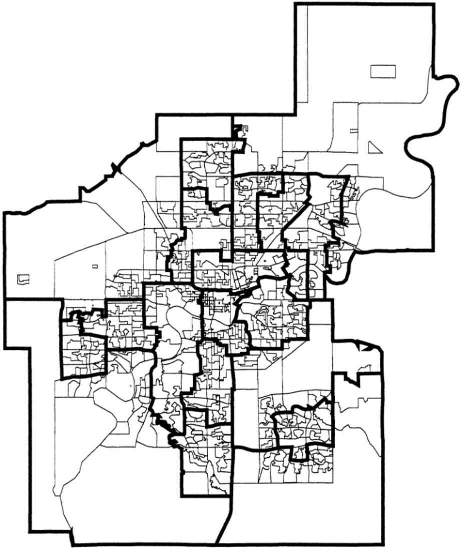


Example: Fleischmann & Paraschi (1988)

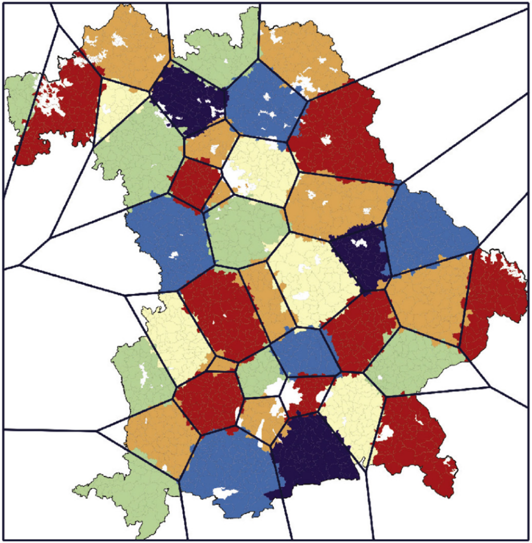
70000 retailers allocated to 168 agents, Germany.

Example: Bozkaya et al. (2011)

828 enumeration areas grouped into 19 political districts, Edmonton.



399 municipalities grouped into 83 microregions, healthcare, Paraná.

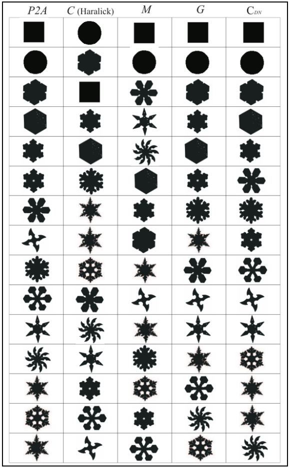


- Balance of attributes
  - e.g. population, resources, workload, ...
- Compactness of zones
  - Shape (e.g. roundness, convexity), distances
- Connectivity of zones

Related but different:

- Clustering problems: usually unconstrained
- Location problems: concerned with placement of facilities

- No clear definition in the literature



(Montero & Bribiesca 2009)

A basic model (Hess et al. 1965)

- Set of *districts* I, $p = |I|$ is the number of required districts.
- Set of *basic units* J, $n = |J|$.
- Attributes a_j, distances d_{ij}, $i \in I$, $j \in J$.
- Variables

$$x_{ij} = \begin{cases} 1, & \text{basic unit } j \in J \text{ is part of district } i \in I, \\ 0, & \text{otherwise.} \end{cases}$$

A basic model (Hess et al. 1965)

$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{(i,j)\in I\times J} d_{ij}x_{ij}, & \\
\textbf{subject to} \quad & \sum_{i\in I} x_{ij} = 1, & \forall j \in J, \\
& \sum_{i\in I} x_{ii} = p, & \forall i \in I, \\
& (1-\tau)\bar{a}x_{ii} \le \sum_{j\in J} a_j x_{ij} \le (1+\tau)\bar{a}x_{ii}, & \forall i \in I, \\
& x_{ij} \in \{0,1\}, & \forall (i,j) \in I \times J.
\end{aligned}
$$

Here $\bar{a} = \sum_{j\in J} a_j/p$ is the *average* attribute, and $\tau \ge 0$ a tolerance.

Variations 1: multiple attributes

- Set of attributes K.
- Attributes a_{kj}, $j \in J$, $k \in K$.

$$(1-\tau_k)\bar{a}_k x_{ii} \leq \sum_{j \in J} a_{kj} x_{ij} \leq (1+\tau_k)\bar{a}_k x_{ii}, \quad \forall i \in I, k \in K$$

- Total distance to centers (p-median)

$$\text{min.} \quad \sum_{(i,j)\in I\times J} d_{ij}x_{ij}$$

- Attribute moments

$$\text{min.} \quad \sum_{(i,j)\in I\times J} a_j d_{ij}x_{ij}$$

$$\text{min.} \quad \sum_{(i,j)\in I\times J} a_j d_{ij}^2 x_{ij}$$

- Maximum distance to centers (p-center)

$$\text{min.} \quad \max_{(i,j)\in I\times J} d_{ij}x_{ij}$$

- Maximum diameter

$$\text{min.} \quad \max_{(j,k)\in J\times J} d_{jk}x_{ij}x_{ik}$$

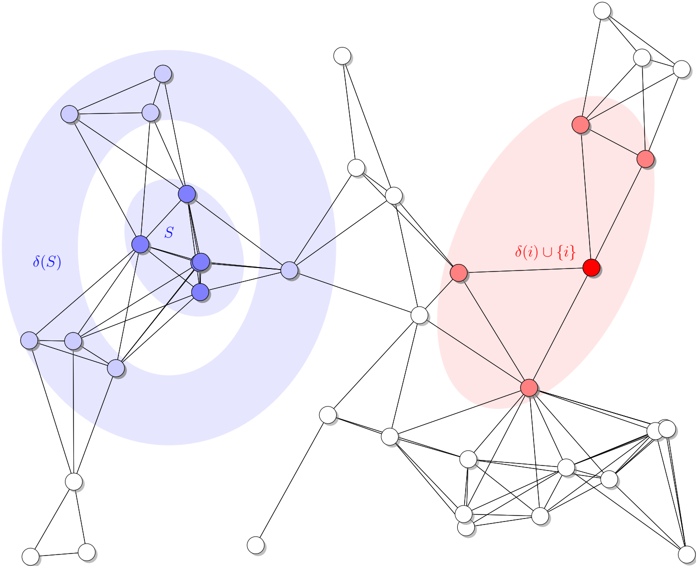
Variations 3: connectivity

- Introduce a graph $G = (J, A)$ over basic units J.
- Require districts to be *connected components* of G.
- Formulation: similar to subtour elimination in TSP
  - Miller-Tucker-Zemlin, flow-based, cutsets
  - Most common (and effective) subsets

$$x_i(\delta(S)) \geq 1 - |S| + x_i(S), \quad \forall i \in I, S \subseteq V \setminus (\delta(i) \cup \{i\}) \tag{Conn}$$

where $x_i(S) = \sum_{j \in S} x_{ij}$ and $\delta(S)$ is the set of all neighbors of S not in S.

Variations 3: connectivity



- Location-allocation (Hess et al. 1965)
  - Heuristic, based on (auxiliary) centers
- Set covering formulations solved by column generation (Mehrotra et al. 1998)
- Repeated solution of the mathematical model (Salazar-Aguilar et al. 2011)

Currently limited to a few hundred basic units.

Location-Allocation (Hess et al. 1965)

1. Define initial p district centers.
2. Repeat until convergence:
    - Solve allocation subproblem: assign each basic unit to some center.
    - Solve location subproblem: define updated district centers.

Location subproblem: find the center of each district in $O((n/p)^2)$.

Let $C \subseteq I$ be the set of current centers:

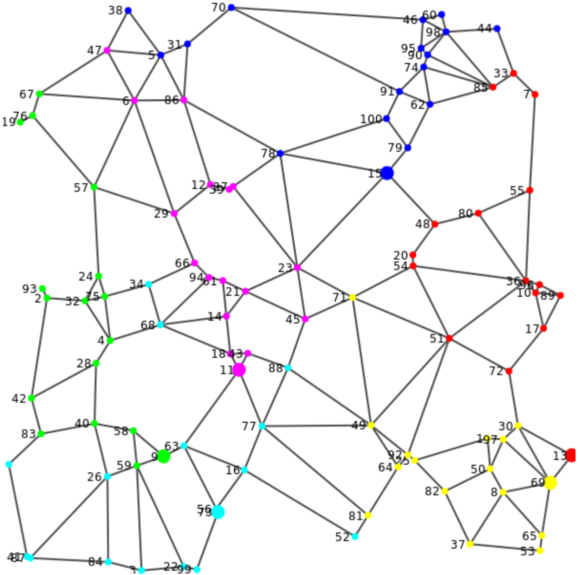
$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{(i,j) \in C \times J} d_{ij} x_{ij}, & \\
\textbf{subject to} \quad & \sum_{i \in C} x_{ij} = 1, & \forall j \in J, \\
& (1-\tau)\bar{a} x_{ii} \leq \sum_{j \in J} a_j x_{ij} \leq (1+\tau)\bar{a} x_{ii}, & \forall i \in C, \\
& x_{ij} \in \{0,1\}, & \forall (i,j) \in C \times J.
\end{aligned}
$$

Reduces number of variables from n^2 to np.

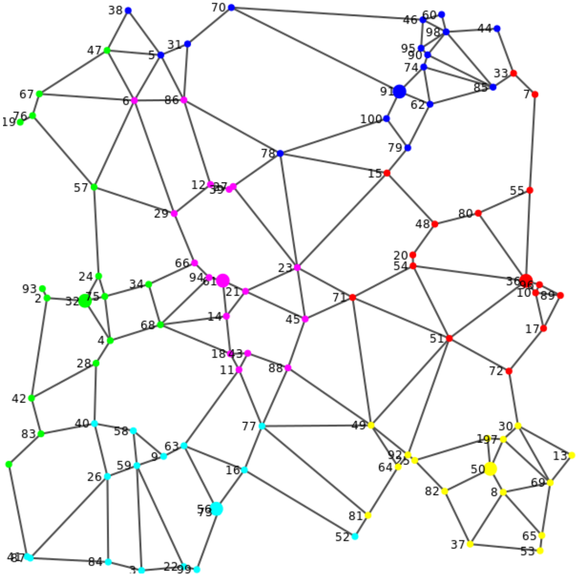
Location-Allocation: solution of allocation subproblem

1. Solve relaxed problem
   - Relax to linear program.
   - Solve for exact balance ($\tau = 0$).
   - Resulting problem is a transportation problem.
2. Split resolution
   - Number of fractional assignments: at most $2(p-1)$.
   - Number of *split units*: at most $p-1$.
   - Split resolution problem: find assignment s.t. balance minimal is NP-hard.

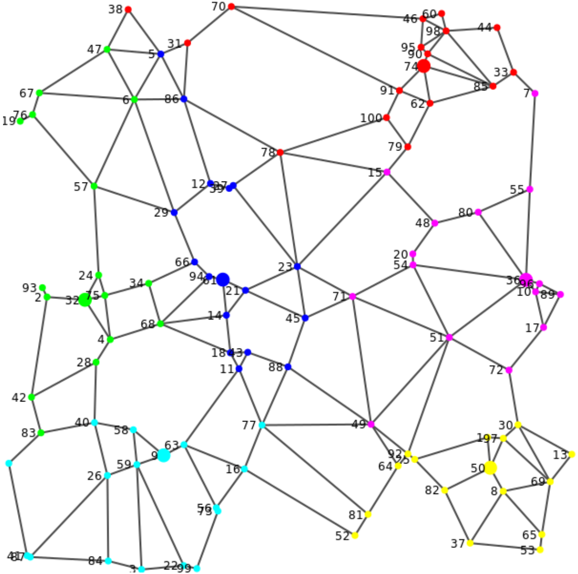
Location-allocation: Example, iteration 1



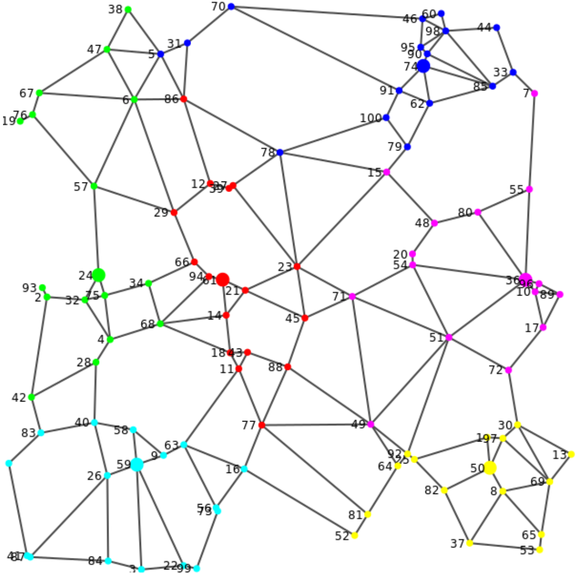
Location-allocation: Example, iteration 2



Location-allocation: Example, iteration 3



Location-allocation: Example, iteration 4



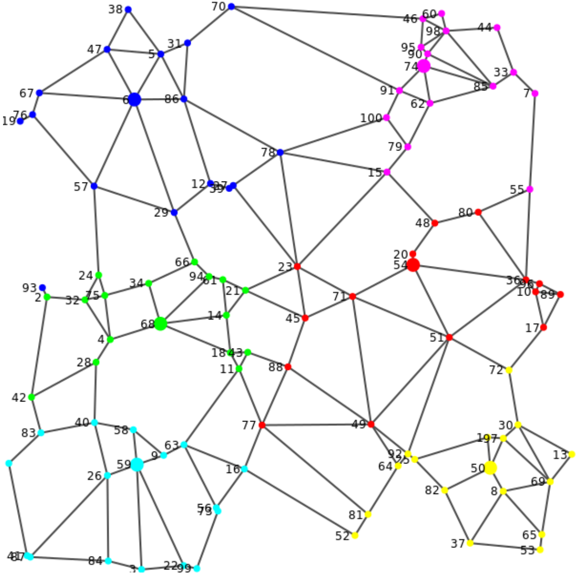
Branch-and-bound-and-cut (Salazar-Aguilar et al. 2011)

- Idea: solve ILP model, add connectivity cuts

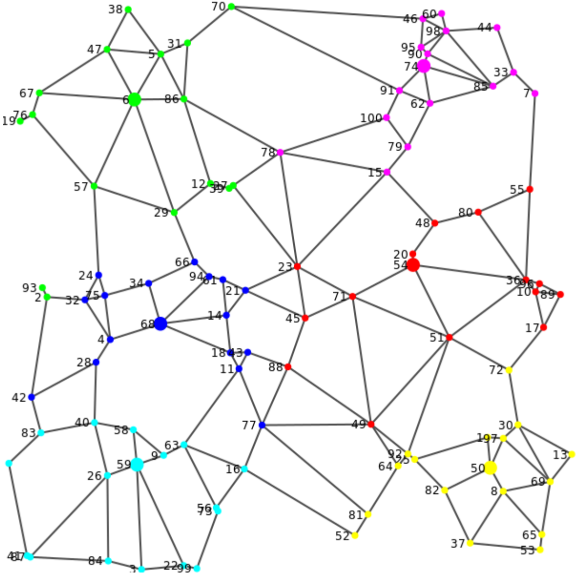
Algorithm:

1. Repeat
   - Solve the current ILP model.
   - If there are unconnected districts: seperate and add restrictions Conn.
2. Until all districts are connected

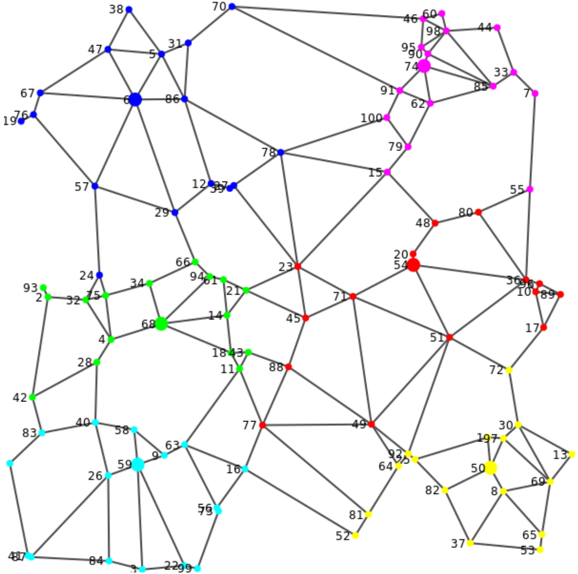
Branch-and-bound-and-cut: Example, iteration 1



Branch-and-bound-and-cut: Example, iteration 2



Branch-and-bound-and-cut: Example, iteration 3



- Proposed by Garfinkel & Nemhauser (1970), evolved by Mehrotra et al. (1998).
- Define the set of possible districts D

$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{i \in D} c_i x_i, & \\
\textbf{subject to} \quad & \sum_{i \in D} [j \in D] x_i = 1, & \forall j \in J, \\
& \sum_{i \in D} x_i = p, & \\
& x_j \in \{0, 1\}, & \forall (j) \in J.
\end{aligned}
$$

- Linear relaxation can be solved by column generation.

- Find balanced and connected districts of negative reduced cost
- Fix a center c, let $J^- = J \setminus \{c\}$.
- For dual variables π_j, $j \in J$ and σ we have

$$\begin{aligned} \textbf{minimize} \quad & -\sigma - \pi_c - \sum_{j \in J^-} (d_{cj} - \pi_j) y_j, \\ \textbf{subject to} \quad & \text{(balance)}, \\ & \text{(connectivity)}, \\ & y_j \in \{0,1\}, \qquad \forall j \in J^-. \end{aligned}$$

Complete branch-and-price algorithm

- Initial solution: greedy construction
  - Fix a reference node u
  - Seed of district: node v of largest distance to u, break ties by largest populaton
  - Grow district: node closest to v, break ties by highest connectivity to district
  - Stop when lower bound has been reached
- Ryan-Foster (same/different) branching
  - Choose most fractional district S_1, most populated unit i in it
  - Choose most fractional district $S_2 \neq S_1$, with $i \in S_2$
  - Choose unit $j \in S_1 \oplus S_2$
  - Creates branches "same(i,j)" and "different(i,j)".
- Depth-first search
  - Prefers "same" branches

Summary and outlook

- Very few exact approaches to territory design
- State of the art limited to small problem sizes (about 200 basic units 10 districts)
- Challenge: push the tractable problem size to 1000 basic units.