LINEAR ALGEBRA

Integer linear programming: formulations, techniques and applications.

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“Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.”¹

Michael F. Atiyah.

1 Vectors
2 Matrices
3 Linear systems
4 Exercises
Finite sequence of real values:

\[ v \in \mathbb{R}^n, \text{ where } n \in \mathbb{N}^+ \]
Vector with all the elements equal to zero (null):

\[ v = [0, 0, \ldots, 0]^t \]
Vectors

Special vectors. Unit

Vector with one element equals to 1 and the rest equal to 0:

\[ \mathbf{v} = [0, \ldots, 0, 1, 0, \ldots, 0]^t \]

* unit vectors are those whose norm is equals to 1
Vectors

Operations. Sum

Sum the elements at the same position:

\[ \mathbf{v}, \mathbf{w} \in \mathbb{R}^n : \]

\[ \mathbf{v} + \mathbf{w} = [v_1, v_2, \ldots, v_n]^t + [w_1, w_2, \ldots, w_n]^t \]

\[ \mathbf{v} + \mathbf{w} = [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]^t \]
Multiplying each element of the vector by the scalar (real number):

\[ \lambda \in \mathbb{R}, v \in \mathbb{R}^n : \]

\[ \lambda \times v = \lambda \times [v_1, v_2, \ldots, v_n]^t \]

\[ \lambda \times v = [\lambda \times v_1, \lambda \times v_2, \ldots, \lambda \times v_n]^t \]
Multiply the elements at the same position and sum the multiplication results:

\[ v, w \in \mathbb{R}^n : \]

\[ v^t \times w = [v_1, v_2, \ldots, v_n] \times \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \]

\[ v^t \times w = \sum_{i=1}^{n} (v_i \times w_i) \]

* the result of the internal product is a scalar.
Given $m$ vectors:

$v^1, v^2, \ldots, v^m \in \mathbb{R}^n$.

A linear combination of them is:

$$\lambda_1 \times v^1 + \lambda_2 \times v^2 + \ldots + \lambda_m \times v^m = \sum_{i=1}^{m} (\lambda_i \times v^i).$$

Where, $\lambda_1, \lambda_2, \ldots, \lambda_m$ are scalars:

$$\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}.$$
Linear combination

\[ \lambda_1 \times \begin{pmatrix} v_1^1 \\ v_2^1 \\ \vdots \\ v_n^1 \end{pmatrix} + \lambda_2 \times \begin{pmatrix} v_1^2 \\ v_2^2 \\ \vdots \\ v_n^2 \end{pmatrix} + \ldots + \lambda_m \times \begin{pmatrix} v_1^m \\ v_2^m \\ \vdots \\ v_n^m \end{pmatrix} = \sum_{i=1}^{m} \lambda_i \times v_i \]
If the sum of the scalars in a linear combination is equal to 1, then the combination is an **AFFINE** combination.

Affine combinations are linear combinations where:

$$\sum_{i=1}^{m} \lambda_i = 1.$$
If the scalars in a linear combination are non-negative, then the combination is a \textbf{CONICAL} combination.

Conical combinations are linear combinations where:

$$\lambda_i \in \mathbb{R}_+, \quad \forall i \in \{1, 2, \ldots, m\}.$$
If a linear combination is affine and conical, then the combination is a **CONVEX** combination.

Convex combinations are linear combinations where:

\[
\sum_{i=1}^{m} \lambda_i = 1
\]

\[
\lambda_i \in \mathbb{R}_+, \quad \forall i \in \{1, 2, \ldots, m\}.
\]
$S \subseteq \mathbb{R}^n$ is a **LINEAR SUBSPACE** if for any two vectors $v, w \in S$ and any two scalars $\alpha, \beta \in \mathbb{R}$, the vector resulting from the linear combination $\alpha \times v + \beta \times w$ is in $S$. 
$S \subseteq \mathbb{R}^n$ is an **AFFINE SUBSPACE** if for any two vectors $v, w \in S$ and any two scalars $\alpha, \beta \in \mathbb{R}$, where $\alpha + \beta = 1$, the vector resulting from the affine combination $\alpha \times v + \beta \times w$ is in $S$.

* Every linear subspace is an affine subspace. However, the opposite does not hold true.
$S \subseteq \mathbb{R}^n$ is a **CONICAL SET** if for any two vectors $v, w \in S$ and any two non-negative scalars $\alpha, \beta \in \mathbb{R}_+$, the vector resulting from the conical combination $\alpha \times v + \beta \times w$ is in $S$.

* Every linear subspace is a conical set. However, the opposite does not hold true.
$S \subseteq \mathbb{R}^n$ is a **CONVEX SET** if for any two vectors $v, w \in S$ and any two non-negative scalars $\alpha, \beta \in \mathbb{R}_+$, where $\alpha + \beta = 1$, the vector resulting from the convex combination $\alpha \times v + \beta \times w$ is in $S$.

* Every conical, affine or linear subspace is a convex set. However, the opposite does not hold true.
The vectors $v^1, v^2, ..., v^m \in \mathbb{R}^n$ are LINEARLY INDEPENDENT if there exists only one linear combination resulting in the null vector, and is when all the scalars of the combination are equal to zero, i.e.:

$$\sum_{i=1}^{m} (\lambda_i \times v^i) = [0, 0, ..., 0]^t \Rightarrow \lambda_i = 0, \forall i \in \{1, 2, ..., m\}$$
The vectors \( v^1, v^2, \ldots, v^m \in \mathbb{R}^n \) are **AFFINELY INDEPENDENT** if:

\[
\sum_{i=1}^{m} (\lambda_i \times v^i) = [0, 0, \ldots, 0]^t
\]

\[
\sum_{i=1}^{m} \lambda_i = 0
\]

\[
\Rightarrow \quad \lambda_i = 0, \quad \forall i \in \{1, 2, \ldots, m\}
\]

* Any set of linearly independent vectors is affine independent. However, the opposite does not hold true.
The **LINEAR SPAN** (also called **LINEAR HULL**) of a set $S \subseteq \mathbb{R}^n$ is the subspace containing all the linear combinations of the elements of $S$:

$$\text{span}(S) = \left\{ \sum_{i=1}^{m} (\lambda_i \times v^i) \mid m \in \mathbb{N}, v^i \in S, \lambda_i \in \mathbb{R} \right\}$$
The **AFFINE SPAN** (also called **AFFINE HULL**) of a set $S \subseteq \mathbb{R}^n$ is the subspace containing all the affine combinations of the elements of $S$:

$$\text{aff}(S) = \left\{ \sum_{i=1}^{m} (\lambda_i \times v^i) \mid m \in \mathbb{N}, v^i \in S, \lambda_i \in \mathbb{R}, \sum_{i=0}^{m} \lambda_i = 1 \right\}$$
The **CONICAL HULL** of a set $S \subseteq \mathbb{R}^n$ is the set containing all the conical combinations of the elements of $S$:

$$
cone(S) = \left\{ \sum_{i=1}^{m} (\lambda_i \times v^i) \mid m \in \mathbb{N}, v^i \in S, \lambda_i \in \mathbb{R}_+ \right\}
$$
The **CONVEX HULL** of a set $S \subseteq \mathbb{R}^n$ is the set containing all the convex combinations of the elements of $S$:

$$convex(S) = \left\{ \sum_{i=1}^{m} (\lambda_i \times v^i) \mid m \in \mathbb{N}, v^i \in S, \lambda_i \in \mathbb{R}_+ , \sum_{i=0}^{m} \lambda_i = 1 \right\}$$
A set $S \subset \mathbb{R}^n$ is a **BASIS** of $\mathbb{R}^n$, if the elements of $S$ are linearly independent and the linear span of $S$ is $\mathbb{R}^n$.

**Remark.** If $S$ is a basis of $\mathbb{R}^n$, then for any vector $v \in \mathbb{R}^n$, there exists exactly one linear combination of the vectors of $S$ equals to $v$. 
Consider a basis $S$ of $\mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$.

If the vector $w \in S$ is multiplied by a scalar $\lambda \neq 0$ in the linear combination of the elements of $S$ that results in $v$, then $(S \setminus \{w\}) \cup \{v\}$ is a basis of $\mathbb{R}^n$. 
Matrices
### Matrices

#### Definition

Rectangular array of real values:

\[
\begin{bmatrix}
\text{value 1, 1} & \text{value 1, 2} & \ldots & \text{value 1, } n \\
\text{value 2, 1} & \text{value 2, 2} & \ldots & \text{value 2, } n \\
\ldots & \ldots & \ldots & \ldots \\
\text{value } m, 1 & \text{value } m, 2 & \ldots & \text{value } m, n
\end{bmatrix}
\]

\[A_{m \times n} \in \mathbb{R}^{m \times n}, (m, n \in \mathbb{N}^+)\]
$a_i \in \mathbb{R}^n$ is the $i$-th row of the matrix $A_{m \times n}$ ($1 \leq i \leq m$):
$a_{.j} \in \mathbb{R}^m$ is the $j$-th column of the matrix $A_{m \times n}$ ($1 \leq j \leq n$):

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<td>$\text{value } m ,, 1$</td>
<td>$\ldots$</td>
<td>$\text{value } m ,, j$</td>
<td>$\ldots$</td>
<td>$\text{value } m ,, n$</td>
</tr>
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</table>
Each column of the matrix is a null vector (i.e. all elements are equal to zero):

\[
\begin{array}{cccccc}
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
\end{array}
\]
Square matrix \((m = n)\), where \(a_{.j}\) is the unit vector where the value 1 occurs at position \(j\) \((1 \leq j \leq n)\) (i.e. all elements are equal to zero except for the main diagonal, whose elements are equal to one):

\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

\[I_{n \times n} \in \mathbb{R}^{n \times n}\]
A square matrix is **LOWER TRIANGULAR** if all the values above the main diagonal are equal to zero (i.e. if \( i < j \), then \( a_{i,j} = 0 \)):

\[
\begin{array}{cccccc}
 a_{1,1} & 0 & 0 & \cdots & 0 \\
 a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\
 a_{3,1} & a_{3,2} & a_{3,3} & \cdots & \cdots \\
 \vdots & \vdots & \ddots & \vdots & 0 \\
 a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n}
\end{array}
\]
A square matrix is **UPPER TRIANGULAR** if all the values below the main diagonal are equal to zero (i.e. if \( i > j \), then \( a_{i,j} = 0 \)):

\[
\begin{array}{cccccc}
  a_{1,1} & a_{1,2} & a_{1,3} & \ldots & a_{1,n} \\
  0 & a_{2,2} & a_{2,3} & \ldots & a_{2,n} \\
  0 & 0 & a_{3,3} & \ddots & \vdots & \ddots \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & \ldots & \ldots & 0 & a_{n-1,n} \\
  0 & 0 & \ldots & 0 & 0 & a_{n,n}
\end{array}
\]
## Matrices

### Operations. Scalar multiplication

Multiply each element of the matrix by the scalar:

\[
\begin{array}{cccc}
\lambda \times a_{1,1} & \lambda \times a_{1,2} & \ldots & \lambda \times a_{1,n} \\
\lambda \times a_{2,1} & \lambda \times a_{2,2} & \ldots & \lambda \times a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda \times a_{m,1} & \lambda \times a_{m,2} & \ldots & \lambda \times a_{m,n} \\
\end{array}
\]

\[\lambda \in \mathbb{R}, A_{m \times n} \in \mathbb{R}^{m \times n} : \]

\[\lambda \times A \in \mathbb{R}^{m \times n}\]
Operations. Scalar sum

Sum each element of the matrix with the scalar:

$$\lambda \in \mathbb{R}, A_{m \times n} \in \mathbb{R}^{m \times n} : \lambda + A \in \mathbb{R}^{m \times n}$$
Operations. Sum

Sum the elements at the same position (row and column):

\[
\begin{array}{cccc}
  a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \ldots & a_{1,n} + b_{1,n} \\
  a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \ldots & a_{2,n} + b_{2,n} \\
  \ldots & \ldots & \ldots & \ldots \\
  a_{m,1} + b_{m,1} & a_{m,2} + b_{m,2} & \ldots & a_{m,n} + b_{m,n}
\end{array}
\]

\[
A_{m \times n}, B_{m \times n} \in \mathbb{R}^{m \times n} : \\
A + B \in \mathbb{R}^{m \times n}
\]
The matrix multiplication $A \times B$, where $A_{m \times n} \in \mathbb{R}^{m \times n}$ and $B_{p \times q} \in \mathbb{R}^{p \times q}$, can be calculated iff $n = p$ (i.e. the number of columns of the first matrix must be equal to the number of lines of the second one).

The resulting matrix $C = A \times B$ satisfies: $C \in \mathbb{R}^{m \times q}$ and the element $c_{ij}$ is equal to the internal product of row $a_i$ of $A$ by column $b_j$ of $B$ ($1 \leq i \leq m$ and $1 \leq j \leq q$).

* Matrix multiplication is not a symmetric operation.
Matrices

Operations. Multiplication

\[ A_{m \times n} \in \mathbb{R}^{m \times n}, \quad B_{n \times q} \in \mathbb{R}^{n \times q} : \]

\[
\begin{array}{cccc}
\sum_{i=1}^{n} a_{1,i} \times b_{i,1} & \sum_{i=1}^{n} a_{1,i} \times b_{i,2} & \cdots & \sum_{i=1}^{n} a_{1,i} \times b_{i,q} \\
\sum_{i=1}^{n} a_{2,i} \times b_{i,1} & \sum_{i=1}^{n} a_{2,i} \times b_{i,2} & \cdots & \sum_{i=1}^{n} a_{2,i} \times b_{i,q} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{n} a_{m,i} \times b_{i,1} & \sum_{i=1}^{n} a_{m,i} \times b_{i,2} & \cdots & \sum_{i=1}^{n} a_{m,i} \times b_{i,q}
\end{array}
\]

\[ A \times B \in \mathbb{R}^{m \times q} \]
Given the matrix $A \in \mathbb{R}^{m \times n}$, the transpose is $A^t \in \mathbb{R}^{n \times m}$, where
the column $a_{.j}$ of $A$ is equal to the row $a^t_{.j}$ of $A^t$ ($1 \leq j \leq n$).

If $A = A^t$ then $A$ is **SYMMETRIC** and if $A = -A^t$ then $A$ is **ANTISYMMETRIC**.
A square matrix $A_{n \times n}$ is **INVERTIBLE** (also known as *non-singular* or *non-degenerate*) if there exists a matrix $B_{n \times n}$ (INVERSE), such that the product of $A$ and $B$ is equal to the identity:

$$A \times B = I_{n \times n}$$

The **INVERSE** of $A$ is denoted by $A^{-1}$.

* $A_{n \times n}$ is invertible iff the lines of $A$ are linearly independent.
Each square matrix $A_{n \times n} \in \mathbb{R}^{n \times n}$ has associated a scalar number named **DETERMINANT** ($\det(A)$ or $|A|$).

$$\det(A) = |A| = \sum_{\pi \in \mathcal{P}_n} \text{sign}(\pi) \prod_{i=1}^{n} a_{i\pi_i}.$$ 

Where, $\mathcal{P}_n$ is the set of all permutations of the elements in $\{1, \ldots, n\}$ and $\text{sign}(\pi)$ is equal to $(-1)^k$, being $k$ the number of two elements swap required to reorder $\pi$.

* The determinant of a $n \times n$ matrix can be computed within $\mathcal{O}(n^{2.373})$ complexity time.
The **RANK** of a matrix $A_{m \times n} \in \mathbb{R}^{m \times n}$ is an integer value equals to the maximum number of matrix rows that are linearly independent.

$$\text{rank}(A) = \max_{L \subseteq \{a_i \mid 1 \leq i \leq n\}} |L|.$$  

and $L$ is linearly independent

* if $A$ is a $n \times n$ matrix and $\det(A) \neq 0$, then $\text{rank}(A) = n$. 
Linear systems
A **LINEAR SYSTEM** receives a matrix $A_{m \times n} \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$, and seeks for a vector $x \in \mathbb{R}^n$, such that $A \times x = b$. 

$$
\begin{array}{cccc}
 a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\
 a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{m,1} & a_{m,2} & \ldots & a_{m,n} \\
\end{array} 
\times 
\begin{array}{c}
 x_1 \\
 x_2 \\
 \vdots \\
 x_n \\
\end{array} 
= 
\begin{array}{c}
 b_1 \\
 b_2 \\
 \vdots \\
 b_m \\
\end{array}
$$
A vector \( x \in \mathbb{R}^n \) is a solution of a linear system \( A_{m \times n} \times x = b \) iff \( x \) is a solution of \( A' \times x = b' \), where \( A' \), \( b' \) are obtained by swapping, respectively, the lines \( a_i \) and \( a_j \) in \( A \) and the entries \( b_i \) and \( b_j \) in \( b \) \((1 \leq i, j \leq m)\).
Elementary operations. Swap lines

Equivalent system obtained by swapping $a_i$ with $a_j$ and $b_i$ with $b_j$. 

\[
\begin{bmatrix}
  a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{i,1} & a_{i,2} & \ldots & a_{i,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{j,1} & a_{j,2} & \ldots & a_{j,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m,1} & a_{m,2} & \ldots & a_{m,n} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{bmatrix}
\times
\begin{bmatrix}
b_1 \\
  b_2 \\
  \vdots \\
  b_m \\
\end{bmatrix}
= 
\begin{bmatrix}
x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{bmatrix}
\begin{bmatrix}
b_i \\
  b_j \\
  \vdots \\
  b_m \\
\end{bmatrix}
A vector $x \in \mathbb{R}^n$ is a solution of a linear system $A_{m \times n} \times x = b$ iff $x$ is a solution of $A' \times x = b'$, where $A'$, $b'$ are obtained by multiplying the line $a_i$ in $A$ and the entry $b_i$ in $b$ by a scalar $\lambda \neq 0$ ($1 \leq i \leq n$).
A vector $x \in \mathbb{R}^n$ is a solution of a linear system $A_{m \times n} \times x = b$ iff $x$ is a solution of $A' \times x = b'$, where $A'$, $b'$ are obtained by replacing the line $a_i$ in $A$ and the entry $b_i$ in $b$ respectively by $a_i + \lambda a_j$ and $b_i + \lambda b_j$ ($1 \leq i, j \leq n$).
A square matrix $P_{m \times m} \in \mathbb{R}^{m \times m}$ is a PERMUTATION MATRIX if it is obtained by the elementary operations over the identity matrix $I_{m \times m}$.

Given a permutation matrix $P_{m \times m}$, a vector $x \in \mathbb{R}^n$ is a solution of a linear system $A_{m \times n} \times x = b$ iff $x$ is a solution of $A' \times x = b'$, where $A' = P \times A$ and $b' = P \times b$. 
Given a linear system $A \times x = b$, apply elementary operations to obtain an equivalent system $A' \times x = b'$, where $A'$ is upper triangular and the elements of the diagonal are equal to 1. Then, use back substitution to find the entries of $x$. 
Given a linear system \( A \times x = b \) if \( A \) is invertible, then \( x = A^{-1} \times b \)

\[ A \times a^{-1} \cdot i = I \cdot i, \] where \( I \cdot i \) is the \( i^{th} \) column of the identity. Thus, the \( i^{th} \) of \( A^{-1} \) can be computed by solving \( A \times x = I \cdot i \).
Given a linear system $A \times x = b$ if $A$ is invertible, then

$$x_i = \frac{\text{det}(B_i)}{\text{det}(A)}$$

where $B_i$ is obtained by replacing in $A$ the column $a_{.,i}$ with $b$. 

Solving linear systems. Cramer’s rule
Given a linear system $A_{m \times n} \times x = b$, consider the matrix $(A, b)_{m \times (n+1)}$, whose first $m$ rows are the rows from $A$ and the last row is the vector $b$. Then:

- If $\text{rank}(A, b) > \text{rank}(A)$, then the linear system has no solution.
- If $\text{rank}(A, b) = \text{rank}(A) = n$, then the linear system has exactly one solution.
- If $\text{rank}(A, b) = \text{rank}(A) < n$, then the linear system has an infinite number of solutions.
Exercises
Exercise 1.

Given $S \subseteq \mathbb{R}^n$ demonstrate that:

$S$ is a linear subspace $\Rightarrow$ $S$ is an affine subspace.
Exercises

Exercise 2.

Demonstrate that:

If \( S \subseteq \mathbb{R}^n \) is a convex set, not necessarily \( S \) is an affine subspace.
Exercise 3.

Given the set of vector \( S \subseteq \mathbb{R}^n \) Demonstrate that:

- If the vectors of \( S \) are linearly independent, then they are also affine independent.
- If the vector of \( S \) are affine independent, not necessarily they are linear independent.
Exercise 4.

Demonstrate that:

If $S \subseteq \mathbb{R}^n$ is a basis of $\mathbb{R}^n$, then for any vector $v \in \mathbb{R}^n$ there exists exactly one linear combination of the vectors of $S$ equals to $v$. 
Exercise 5.

Demonstrate the *change of basis* property.
Exercise 6.

Suppose are given two matrices $A$ and $B$, such that it is possibly to multiply $A \times B$ and also $B \times A$. Demonstrate that not necessarily $A \times B = B \times A$. 
Exercise 7.

Demonstrate that:

1. $\left( A^t \right)^t = A$.
2. $\left( A + B \right)^t = A^t + B^t$.
3. $\left( A \times B \right)^t = B^t \times A^t$. 
Exercise 8.

Given an invertible matrix $A \in \mathbb{R}^{n \times n}$, demonstrate that:

- $A^{-1}$ is unique, invertible and $(A^{-1})^{-1} = A$.
- $A^t$ is invertible and $(A^t)^{-1} = (A^{-1})^t$.
- If $B \in \mathbb{R}^{n \times n}$ is invertible, then $A \times B$ is invertible and $(A \times B)^{-1} = B^{-1} \times A^{-1}$. 
Exercise 9.

Demonstrate that:

A square matrix $A$ is invertible iff $\det(A) \neq 0$. 
Linear algebra

Integer linear programming: formulations, techniques and applications.

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