LINEAR PROGRAMMING: DUAL PROBLEM AND COMPLEMENTARY SLACKNESS

Integer linear programming: formulations, techniques and applications.

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“Duality is one of the oldest and most fruitful ideas in Mathematics.”

Michael F. Atiyah.

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1. Previous lecture
2. Dual problem
3. Duality theorems
4. Complementary slackness
5. Exercises
Previous lecture
Canonical problem

\[ CLP(c, A, b): \]

\[
\min \quad c^t \times x \\
\text{s.t.:} \quad A \times x = b \\
\quad x \in \mathbb{R}^n_+ 
\]
General problem

\[
\begin{align*}
LP_{\min}(c, A=, A\leq, A\geq, b=, b\leq, b\geq) \\
(LP_{\max}(c, A=, A\leq, A\geq, b=, b\leq, b\geq)):
\end{align*}
\]

\[
\begin{align*}
\min(\max) & \quad c^t \times x \\
\text{s.t.:} & \\
A= \times x &= b= \\
A\leq \times x &\leq b\leq \\
A\geq \times x &\geq b\geq \\
x &\in \mathbb{R}^n
\end{align*}
\]
Classifications

- INFEASIBLE.
- UNBOUNDED.
- SOLVABLE.
Equivalences

- $\max c^t \times x \equiv \min -c^t \times x$
- $A \times x \geq b \equiv -A \times x \leq -b$
- $A \times x \leq b \equiv (A, I) \times (x, y) = b$
- $A \times x \geq b \equiv (A, -I) \times (x, y) = b$
- $A \times x = b \equiv \begin{cases} A \times x \leq b \\ A \times x \geq b \end{cases}$
- $x_i \in \mathbb{R} \equiv \begin{cases} x'_i, x''_i \in \mathbb{R}_+ \\ x_i = x'_i - x''_i \end{cases}$
Dual problem
Consider a PRIMAL LINEAR PROGRAMMING problem \( PLP(c, A, b) \) compound by two vectors \( c \in \mathbb{R}^n \) and \( b \in \mathbb{R}^m \), and a matrix \( A \in \mathbb{R}^{m \times n} \) \((m, n \in \mathbb{N})\) and formulated as follows (standard form):

\[
\begin{align*}
\text{max} & \quad c^t \times x \\
\text{s.t.:} & \quad A \times x \leq b \\
& \quad x_i \geq 0 \quad \forall 1 \leq i \leq n \\
& \quad x \in \mathbb{R}^n
\end{align*}
\]

* By the equivalences of the previous lecture any linear programming problem can be formulated in the above form.
Given a primal linear programming problem $\text{PLP}(c, A, b)$, the associated **DUAL LINEAR PROGRAMMING** problem $\text{DLP}(c, A, b)$ is formulated as follows:

$$\min b^t \times y$$

s.t.:

$$A^t \times y \geq c$$
$$y_i \geq 0 \quad \forall 1 \leq i \leq m$$
$$y \in \mathbb{R}^m$$
Example. Lunch

Minimize total number of fat milligram in a lunch consisting of a salad and a soup, where the nutritional information is:

<table>
<thead>
<tr>
<th>Vitamin A</th>
<th>Vitamin B</th>
<th>Fats</th>
</tr>
</thead>
<tbody>
<tr>
<td>80 mcg/100g</td>
<td>0.4 mcg/100g</td>
<td>4 mg/100g</td>
</tr>
<tr>
<td>60 mcg/100g</td>
<td>0.2 mcg/100g</td>
<td>6 mg/100g</td>
</tr>
</tbody>
</table>

The nutritional requirements are: at least 450 mcg of vitamin A and 2 mcg of vitamin B, and to avoid consuming more than 700 g.
Example. Lunch Formulation (previous lecture)

\[\begin{align*}
\text{min} & \quad 4 \times x_{\text{salad}} + 6 \times x_{\text{soup}} \\
\text{s.t.:} & \\
80 \times x_{\text{salad}} + 60 \times x_{\text{soup}} & \geq 450 \\
0.4 \times x_{\text{salad}} + 0.2 \times x_{\text{soup}} & \geq 2 \\
x_{\text{salad}} + x_{\text{soup}} & \leq 7 \\
x_{\text{salad}}, x_{\text{soup}} & \geq 0
\end{align*}\]
Example. Lunch
Formulation as PLP

\[
\begin{align*}
\text{min} & \quad 4 \times x_{\text{salad}} + 6 \times x_{\text{soup}} \\
\text{s.t.:} & \quad 80 \times x_{\text{salad}} + 60 \times x_{\text{soup}} \geq 450 \\
& \quad 0.4 \times x_{\text{salad}} + 0.2 \times x_{\text{soup}} \geq 2 \\
& \quad x_{\text{salad}} + x_{\text{soup}} \leq 7 \\
& \quad x_{\text{salad}}, x_{\text{soup}} \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad -4 \times x_{\text{salad}} - 6 \times x_{\text{soup}} \\
\text{s.t.:} & \quad -80 \times x_{\text{salad}} - 60 \times x_{\text{soup}} \leq -450 \\
& \quad -0.4 \times x_{\text{salad}} - 0.2 \times x_{\text{soup}} \leq -2 \\
& \quad x_{\text{salad}} + x_{\text{soup}} \leq 7 \\
& \quad x_{\text{salad}}, x_{\text{soup}} \geq 0
\end{align*}
\]
Example. Lunch
Primal and dual formulations

\[\begin{align*}
\text{max} & \quad -4 \times x_{\text{salad}} - 6 \times x_{\text{soup}} \\
\text{s.t.} : & \quad -80 \times x_{\text{salad}} - 60 \times x_{\text{soup}} \leq -450 \\
& \quad -0.4 \times x_{\text{salad}} - 0.2 \times x_{\text{soup}} \leq -2 \\
& \quad x_{\text{salad}} + x_{\text{soup}} \leq 7 \\
& \quad x_{\text{salad}}, x_{\text{soup}} \geq 0
\end{align*}\]

\[\begin{align*}
\text{min} & \quad -450 \times y_A - 2 \times y_B + 7 \times y_{\text{weight}} \\
\text{s.t.} : & \quad -80 \times y_A - 0.4 \times y_B + y_{\text{weight}} \geq -4 \\
& \quad -60 \times y_A - 0.2 \times y_B + y_{\text{weight}} \geq -6 \\
y_A, y_B, y_{\text{weight}} \geq 0
\end{align*}\]
Each dual variable is related to a nutrient or a measure and they can be interpreted as the amount of milligrams of fat per each unit the nutrient/measure, i.e.:

\( y_A \), amount of milligram of fat for each unit of vitamin \( A \).

\( y_B \), amount of milligram of fat for each unit of vitamin \( B \).

\( y_{weight} \), amount of milligram of fat for each \( 100g \) of food.

The dual seeks for a solution where the amount of fat per nutrient is not greater than the amount of fat per dish:

\[
-80 \times y_A - 0.4 \times y_B + y_{weight} \geq -4 \quad \iff \quad 80 \times y_A + 0.4 \times y_B - y_{weight} \leq 4
\]

\[
-60 \times y_A - 0.2 \times y_B + y_{weight} \geq -6 \quad \iff \quad 60 \times y_A + 0.2 \times y_B - y_{weight} \leq 6
\]
Primal and dual

\[ PLP(c, A, b) : \]
\[
\begin{align*}
\text{max} & \quad c^t \times x \\
\text{s.t.:} & \quad A \times x \leq b \\
& \quad x_i \geq 0 \quad \forall 1 \leq i \leq n \\
& \quad x \in \mathbb{R}^n
\end{align*}
\]

\[ DLP(c, A, b) : \]
\[
\begin{align*}
\text{min} & \quad b^t \times y \\
\text{s.t.:} & \quad A^t \times y \geq c \\
& \quad y_i \geq 0 \quad \forall 1 \leq i \leq m \\
& \quad y \in \mathbb{R}^m
\end{align*}
\]
Primal and dual

**Primal problem (P LP):**

\[ \text{max } \mathbf{c}^T \times \mathbf{x} \]
\[ \text{s.t.:} \]
\[ \mathbf{A} \times \mathbf{x} \leq \mathbf{b} \]
\[ x_i \geq 0 \quad \forall 1 \leq i \leq n \]
\[ x \in \mathbb{R}^n \]

**Dual problem (D LP):**

\[ \text{min } \mathbf{b}^T \times \mathbf{y} \quad (\times - 1) \]
\[ \text{s.t.:} \]
\[ \mathbf{A}^T \times \mathbf{y} \geq \mathbf{c} \quad (\times - 1) \]
\[ y_i \geq 0 \quad \forall 1 \leq i \leq m \]
\[ y \in \mathbb{R}^m \]
Dual problem

Primal and dual

\[
\begin{align*}
\text{PLP}(c, A, b) : & \quad \max \ c^t \times x \\
& \quad \text{s.t.:} \ A \times x \leq b \\
& \quad \quad x_i \geq 0 \quad \forall 1 \leq i \leq n \\
& \quad \quad x \in \mathbb{R}^n \\
\text{DLP}(c, A, b) : & \quad (\text{PLP}(-b, -A^t, -c)) \\
& \quad \max \ -b^t \times y \\
& \quad \text{s.t.:} \ -A^t \times y \leq -c \\
& \quad \quad y_i \geq 0 \quad \forall 1 \leq i \leq m \\
& \quad \quad y \in \mathbb{R}^m
\end{align*}
\]
The dual of $PLP(c, A, b)$ is $PLP(-b, -A^t, -c)$.

Thus, the dual of $PLP(-b, -A^t, -c)$ is $PLP(-(c), -(A^t)^t, -(b)) = PLP(c, A, b)$.

Therefore, the dual of the dual is the primal.
Duality theorems
Theorem. If $x$ is a feasible solution of $PLP(c, A, b)$ and $y$ a feasible solution of $DLP(c, A, b)$, then

$$b^t \times y \geq c^t \times x$$
Weak duality theorem. Proof

Since \( y \) is a feasible solution of \( DLP(c, A, b) \) and \( x \) is a feasible solution of \( PLP(c, A, b) \), it follows

\[
A^t \times y \geq c
\]
Weak duality theorem. Proof

Since \( y \) is a feasible solution of \( DLP(c, A, b) \) and \( x \) is a feasible solution of \( PLP(c, A, b) \), it follows

\[
A^t \times y \geq c \quad (\times x \geq 0)
\]
Duality theorems

Weak duality theorem. Proof

Since $y$ is a feasible solution of $DLP(c, A, b)$ and $x$ is a feasible solution of $PLP(c, A, b)$, it follows

$$A^t \times y \geq c \quad (\times x \geq 0)$$

$$\Rightarrow (A^t \times y)^t \times x \geq c^t \times x$$
Since $y$ is a feasible solution of $\text{DLP}(c, A, b)$ and $x$ is a feasible solution of $\text{PLP}(c, A, b)$, it follows

$$A^t \times y \geq c \quad (\times x \geq 0)$$

$$\Rightarrow \quad (A^t \times y)^t \times x \geq c^t \times x$$

$$\Rightarrow \quad y^t \times A \times x \geq c^t \times x$$
Duality theorems

Weak duality theorem. Proof

Since $y$ is a feasible solution of $DLP(c, A, b)$ and $x$ is a feasible solution of $PLP(c, A, b)$, it follows

\[
A^t \times y \geq c \quad (\times x \geq 0)
\]
\[
\Rightarrow (A^t \times y)^t \times x \geq c^t \times x
\]
\[
\Rightarrow y^t \times A \times x \geq c^t \times x \quad (A \times x \leq b)
\]
Duality theorems

Weak duality theorem. Proof

Since $y$ is a feasible solution of $DLP(c, A, b)$ and $x$ is a feasible solution of $PLP(c, A, b)$, it follows

$$A^t \times y \geq c \quad (\times x \geq 0)$$

$$\Rightarrow (A^t \times y)^t \times x \geq c^t \times x$$

$$\Rightarrow y^t \times A \times x \geq c^t \times x \quad (A \times x \leq b)$$

$$\Rightarrow y^t \times b \geq c^t \times x$$
Weak duality theorem. Proof

Since $y$ is a feasible solution of $DLP(c, A, b)$ and $x$ is a feasible solution of $PLP(c, A, b)$, it follows

$$A^t \times y \geq c \quad (\times x \geq 0)$$

$$\Rightarrow (A^t \times y)^t \times x \geq c^t \times x$$

$$\Rightarrow y^t \times A \times x \geq c^t \times x \quad (A \times x \leq b)$$

$$\Rightarrow y^t \times b \geq c^t \times x$$

$$\Rightarrow b^t \times y \geq c^t \times x$$
Duality theorems

Weak duality theorem. Implications

If $PLP(c, A, b)$ and $DPL(c, A, b)$ have optimal solutions, then:

- Given any feasible solution $y$ of the dual, the value $b^t \times y$ is an **UPPER BOUND** for the optimal value of the primal.

- Given any feasible solution $x$ of the primal, the value $c^t \times x$ is a **LOWER BOUND** for the optimal value of the dual.
Theorem. $x^*$ is an optimal solution of $PLP(c, A, b)$ iff $y^*$ is an optimal solution of $DLP(c, A, b)$, where

$$c^t \times x^* = b^t \times y^*$$
Duality theorems

Strong duality theorem. Proof idea

Proof that if there exists an optimal solution \( x^* \) for \( PLP(c, A, b) \), then there exists a feasible solution \( y \) for \( DLP(c, A, b) \), such that: \( c^t \times x^* = b^t \times y \). \(^2\)

The weak duality implies that such \( y \) is optimal for \( DLP(c, A, b) \).

Since the dual of the dual is the primal, the other direction holds.

\(^2\)The proof can be done by using the Simplex method or can be found in the paper “A short note on strong duality: without Simplex and without theorems of alternatives” by Somdeb Lahiri (2017).
Given a primal linear program $\text{PLP}(c, A, b)$ and its dual $\text{DPL}(c, A, b)$, there are four possibilities:

- The primal and the dual are infeasible.
- The primal is infeasible and the dual is unbounded.
- The primal is unbounded and the dual is infeasible.
- The primal and the dual are feasible and the optimal solution value is the same for both problems.
Complementary slackness
Theorem. Given an optimal solution $x^*$ of $PLP(c, A, b)$ and an optimal solution $y^*$ of $DLP(c, A, b)$, it follows

$$(c - A^t \times y^*)^t \times x^* = 0 \text{ and } (b - A \times x^*)^t \times y^* = 0$$
Any pair \((x, y)\) of (primal, dual) feasible solutions satisfies:

\[ A^t \times y \geq c \]
Complementary slackness theorem. Proof

Any pair \((x, y)\) of (primal, dual) feasible solutions satisfies:

\[
A^t \times y \geq c \quad (\times x \geq 0)
\]

\[
\Rightarrow (A^t \times y)^t \times x \geq c^t \times x
\]
Complementary slackness

Complementary slackness theorem. Proof

Any pair \((x, y)\) of (primal, dual) feasible solutions satisfies:

\[
A^t \times y \geq c \quad (\times x \geq 0)
\]

\[
\Rightarrow (A^t \times y)^t \times x \geq c^t \times x
\]

\[
\Rightarrow y^t \times A \times x \geq c^t \times x
\]
Complementary slackness theorem. Proof

Any pair \((x, y)\) of (primal, dual) feasible solutions satisfies:

\[
A^t \times y \geq c \quad (\times x \geq 0)
\]

\[
\Rightarrow (A^t \times y)^t \times x \geq c^t \times x
\]

\[
\Rightarrow y^t \times A \times x \geq c^t \times x \quad (A \times x \leq b)
\]

\[
\Rightarrow y^t \times b \geq y^t \times A \times x \geq c^t \times x
\]
Any pair \((x, y)\) of (primal, dual) feasible solutions satisfies:

\[
A^t \times y \geq c \quad (\times x \geq 0)
\]

\[
\Rightarrow (A^t \times y)^t \times x \geq c^t \times x
\]

\[
\Rightarrow y^t \times A \times x \geq c^t \times x \quad (A \times x \leq b)
\]

\[
\Rightarrow y^t \times b \geq y^t \times A \times x \geq c^t \times x
\]

Since \((x^*, y^*)\) are optimal (strong duality):

\[
y^{*t} \times b = y^{*t} \times A \times x^* = c^t \times x^*
\]
Complementary slackness theorem. Proof

Any pair \((x, y)\) of (primal, dual) feasible solutions satisfies:

\[
A^t \times y \geq c \quad (\times x \geq 0)
\]

\[
\Rightarrow (A^t \times y)^t \times x \geq c^t \times x
\]

\[
\Rightarrow y^t \times A \times x \geq c^t \times x \quad (A \times x \leq b)
\]

\[
\Rightarrow y^t \times b \geq y^t \times A \times x \geq c^t \times x
\]

Since \((x^*, y^*)\) are optimal (strong duality):

\[
y^{*t} \times b = y^{*t} \times A \times x^* = c^t \times x^*
\]

\[
y^{*t} \times b - y^{*t} \times A \times x^* = 0 \quad 0 = c^t \times x^* - y^{*t} \times A \times x^*
\]
Any pair \((x, y)\) of (primal, dual) feasible solutions satisfies:

\[
A^t \times y \geq c \quad (\times x \geq 0)
\]

\[
\Rightarrow (A^t \times y)^t \times x \geq c^t \times x
\]

\[
\Rightarrow y^t \times A \times x \geq c^t \times x \quad (A \times x \leq b)
\]

\[
\Rightarrow y^t \times b \geq y^t \times A \times x \geq c^t \times x
\]

Since \((x^*, y^*)\) are optimal (strong duality):

\[
y^{*t} \times b = y^{*t} \times A \times x^* = c^t \times x^*
\]

\[
y^{*t} \times b - y^{*t} \times A \times x^* = 0 \quad 0 = c^t \times x^* - y^{*t} \times A \times x^*
\]

\[
(b - A \times x^*)^t \times y^* = 0 \quad 0 = (c - A^t \times y^*)^t \times x^*
\]
Is \((x_{\text{salad}} = 3, x_{\text{soup}} = 4)\) an optimal solution for the Lunch problem?

\[
\begin{align*}
\text{max} & \quad -4 \times x_{\text{salad}} - 6 \times x_{\text{soup}} \\
\text{s.t.} : & \\
-80 \times x_{\text{salad}} - 60 \times x_{\text{soup}} & \leq -450 \\
-0.4 \times x_{\text{salad}} - 0.2 \times x_{\text{soup}} & \leq -2 \\
x_{\text{salad}} + x_{\text{soup}} & \leq 7 \\
x_{\text{salad}}, x_{\text{soup}} & \geq 0
\end{align*}
\]
Given \( (x_{\text{salad}} = 3, x_{\text{soup}} = 4) \), to test the optimality, analyse the constraints and the dual variables:

\[
-80 \times x_{\text{salad}} - 60 \times x_{\text{soup}} \leq -450 \quad (y_A)
\]
\[
-0.4 \times x_{\text{salad}} - 0.2 \times x_{\text{soup}} \leq -2 \quad (y_B)
\]
\[
x_{\text{salad}} + x_{\text{soup}} \leq 7 \quad (y_{\text{weight}})
\]
Replace the values of \((x_{salad} = 3, x_{soup} = 4)\) in the constraints:

\[
\begin{align*}
-80 \times (3) - 60 \times (4) &= -480 < -450 \\
-0.4 \times (3) - 0.2 \times (4) &= -2 \\
(3) + (4) &= 7
\end{align*}
\]

\((y_A = 0)\)  
\((y_B \geq 0)\)  
\((y_{weight} \geq 0)\)
Optimality test via complementary slackness

Analyse the dual constraints for \((y_A = 0, y_B, y_{\text{weight}})\):

\[-80 \times (0) - 0.4 \times y_B + y_{\text{weight}} \geq -4\]
\[-60 \times (0) - 0.2 \times y_B + y_{\text{weight}} \geq -6\]
The dual constraints associated with the non-zero primal variables must satisfy the equality:

\[-80 \times (0) - 0.4 \times y_B + y_{\text{weight}} = -4 \quad (x_{\text{salad}} = 3 \neq 0)\]
\[-60 \times (0) - 0.2 \times y_B + y_{\text{weight}} = -6 \quad (x_{\text{soup}} = 4 \neq 0)\]
The solution of the system is $(y_A = 0, y_B = -10, y_{weight} = -8)$ which is infeasible.

\[-80 \times (0) - 0.4 \times y_B + y_{weight} = -4 \quad (x_{\text{salad}} = 3 \neq 0)\]

\[-60 \times (0) - 0.2 \times y_B + y_{weight} = -6 \quad (x_{\text{soup}} = 4 \neq 0)\]
Complementary slackness

Optimality test via complementary slackness

There are no feasible dual solution \( y \) associated with

\((x_{\text{salad}} = 3, x_{\text{soup}} = 4)\), thus \((x_{\text{salad}} = 3, x_{\text{soup}} = 4)\) cannot be optimal for the primal.
Exercises
Exercise 1. Air transport (dual)

Consider your formulation for the **Air transport** problem of the previous lecture:

a) Give a dual formulation and explain the variables meaning.

b) Select a feasible solution of the primal and test its optimality via complementary slackness.
Consider your formulation for the Cargo ship problem of the previous lecture:

a) Give a dual formulation and explain the variables meaning.

b) Select a feasible solution of the primal and test its optimality via complementary slackness.
Exercises

Exercise 3. Servers communication (dual)

Consider your formulation for the Servers communication problem of the previous lecture:

a) Give a dual formulation and explain the variables meaning.

b) Select a feasible solution of the primal and test its optimality via complementary slackness.
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