

Geometrical Transformations

- 2-D x 3-D
- Linear
- Affine
- Projective
- Matrix Composition

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Linear Transformation

- Satisfies the following conditions (for $u, v \in \mathbb{R}^n, \lambda \in \mathbb{R}$):
 - $L(u+v) = L(u) + L(v)$
 - $L(\lambda u) = \lambda L(u)$
- Examples of linear transformations: **Rotation, Scaling and Shearing**
- Preserve parallel lines

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Affine Transformation

- $A(u)$ is *affine* iff $A(u) = L(u) + c$, where $u \in \mathbb{R}^n$, L is a linear transformation and c is a constant
- Linear transformations are a special case of affine transformations
- Example of affine transform: **Translation**
- Preserve parallel lines

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Matrix Representation

- The left (green) sub-matrix represents a linear transformation (rotation, scaling, shearing)
- The right (yellow) sub-matrix represents translation
- Non-square matrices have no general inverse

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

It has no physical meaning

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Homogeneous Coordinates (2-D)

- Provide a way to represent affine transformations using square matrices
- n-dimensional position vector represented with (n+1) coordinates

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Homogeneous Coordinates

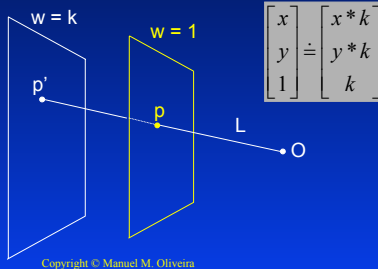
- Operates in projective space (point-line duality)
- The (n+1)th coordinate represents the plane upon which the transformation operates

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Projective Space

- All points along line L are projectively equivalent



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Projective Transformation

- If $[m_{31} \ m_{32} \ m_{33}] = [0 \ 0 \ 1]$ then orthographic projection (affine); otherwise, perspective projection
- Perspective projection only preserve parallel lines if they are parallel to the projection plane

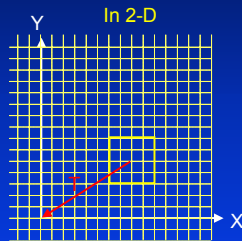
$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Translation

- $x' = x + dx$
- $y' = y + dy$
- $z' = z + dz$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

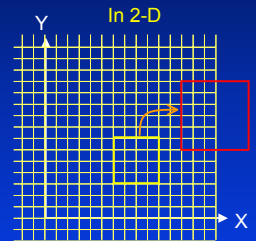


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Scaling

- $x' = S_x x$
- $y' = S_y y$
- $z' = S_z z$

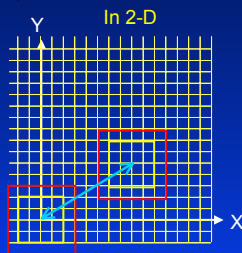
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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Scaling Around a Point

- Translate the point to the origin, then scale and translate the origin back to the point



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Shear (2-D)

- $x' = x + h_x y$
- $y' = y + h_y x$



$$H_x = \begin{bmatrix} 1 & h_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad H_y = \begin{bmatrix} 1 & 0 & 0 \\ h_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad H_{xy} = \begin{bmatrix} 1 & h_x & 0 \\ h_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Shear (3-D)

- The same idea, but with more degrees of freedom
- Most interesting case: shear of x and y along z
- $x' = x + h_x z$
- $y' = y + h_y z$

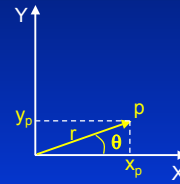


$$H_{xy} = \begin{bmatrix} 1 & 0 & h_x & 0 \\ 0 & 1 & h_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Rotation around the origin (2-D)

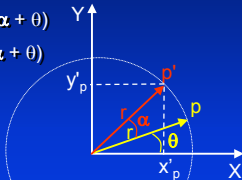
- $x_p = r \cos(\theta)$
- $y_p = r \sin(\theta)$



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Rotation around the origin (2-D)

- $x_p = r \cos(\theta)$
- $y_p = r \sin(\theta)$
- $x'_p = r \cos(\alpha + \theta)$
- $y'_p = r \sin(\alpha + \theta)$

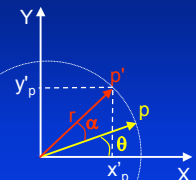


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Rotation around the origin (2-D)

- $x'_p = r \cos(\alpha + \theta) = r (\cos(\alpha) \cos(\theta) - \sin(\alpha) \sin(\theta))$
- $y'_p = r \sin(\alpha + \theta) = r (\sin(\alpha) \cos(\theta) + \sin(\theta) \cos(\alpha))$
- $x'_p = x_p \cos(\alpha) - y_p \sin(\alpha)$
- $y'_p = x_p \sin(\alpha) + y_p \cos(\alpha)$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

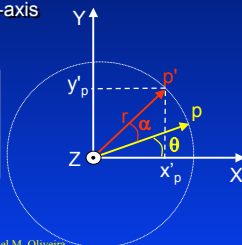


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Rotation (3-D)

- Rotations around the principal axis in 3-D can be derived from rotations in 2-D
- Example: rotation around Z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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Rotation (3-D)

- Rotation around X-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Rotation around Y-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \alpha & 0 & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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Rotation about arbitrary 3-D axis

- Difficult to compute the **Euler** angles that accomplish such rotation
- Easily accomplished using **Quaternions**

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Matrix Composition

- More complex transformations are obtained by multiplying the matrices of the individual transformations
- Matrix multiplication is not commutative ($AB \neq BA$), but it is associative ($(AB)C = A(BC)$)
- Sometimes it is ok to switch the order of transformations:

	T	Sc	R
T	ok	no	no
Sc	—	ok	ok*
R	—	—	ok

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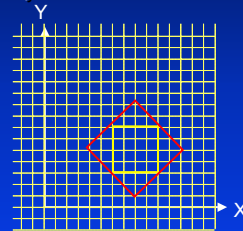
Orthogonal Matrices

- If all rows/columns of a matrix are LI and their lengths are all equal to 1, the matrix is called **orthogonal**
- The inverse of an orthogonal matrix is its **transpose**
- Rotation matrices are orthogonal
- Transformations involving only rotations and translations are called rigid body transformations

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Composite Transformation

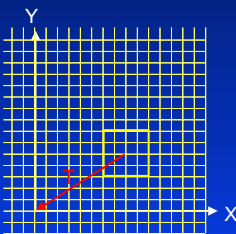
- Rotate the square by 45 degrees around its center and scale by 1.5



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Composite Transformation

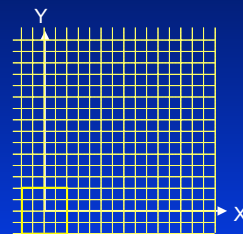
- Translate the center of the square to the origin (T)



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Composite Transformation

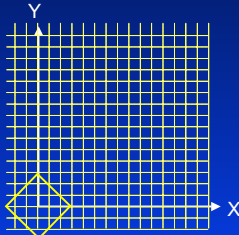
- Translate the center of the square to the origin (T)



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Composite Transformation

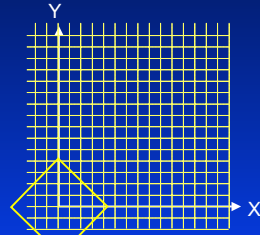
- Rotate (R) (or scale (S))



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Composite Transformation

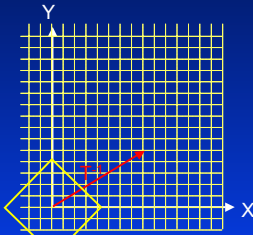
- Scale (S) (or rotate (R))



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Composite Transformation

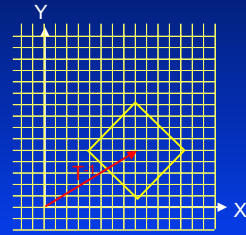
- Translate back (T^{-1})



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Composite Transformation

- Composite: $T^{-1}SRT$ or $T^{-1}RST$



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Composite Transformation

- Composite: $T^{-1}SRT$

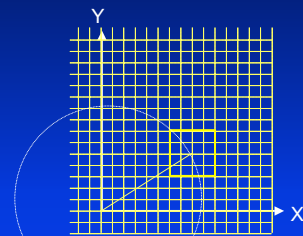
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Note:** In OpenGL one specifies the transformation matrices in the reverse order they should be applied. In this case, one should first inform T^{-1} then S then R , and finally T . They are composite as $((I T^{-1})S)RT$, producing the same result.

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Composite Transformation

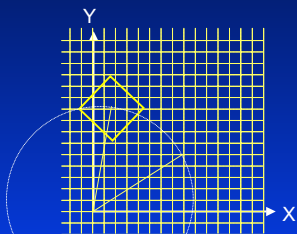
- Without translating to the origin: SR



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Composite Transformation

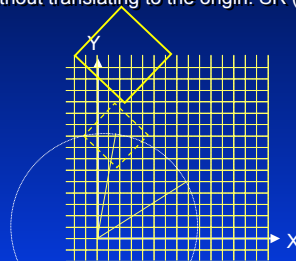
- Without translating to the origin: SR (after the rotation)



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Composite Transformation

- Without translating to the origin: SR (after the scale)



Wrong result!

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