A General Framework for Subspace Detection in Unordered Multidimensional Data

– Supplementary Material A –
Some Models of Geometry and the Geometric Interpretation of Subspaces

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A.1 Description

This document summarizes the geometric primitives that can be represented as weighted k-dimensional oriented subspace (*i.e.,* k-blades) in four models of geometry (MOGs). The Euclidean (Section A.2), homogeneous (Section A.3), conformal (Section A.4), and conic (Section A.5) models are discussed. Recall that a MOG provides a practical geometric interpretation to blades. Such interpretation is achieved by embedding the d-dimensional **base space** $\mathbb{R}^d$ (*i.e.,* space where the geometric interpretation happens) into an n-dimensional **representational space** $\mathbb{R}^n$ (*i.e.,* the total vector space), and by defining a metric to the representational space. The geometric properties of the space depend on the chosen metric. It is important to emphasize that the geometric interpretation of subspaces is not restricted to the examples presented in this document. It can be extended to higher dimensions, to other geometric shapes,

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and to other MOGs. The books by Dorst et al. [1] and Perwass [2] provide in-depth treatments to the subject. See also our tutorial on geometric algebra [3], which provides a quick reference for defining geometric primitives as blades in the discussed MOGs, from parameters that are typically used with linear algebra.

A.2 The Euclidean Model

As the name suggests, in the Euclidean MOG one assumes Euclidean metric for $\mathbb{R}^n$. This way, k-blades are geometrically interpreted as k-dimensional Euclidean subspaces, *i.e.,* oriented **flats** (*e.g.,* straight lines, planes, and their higher-dimensional counterparts) that pass through the origin of the vector space.

Euclidean subspaces in $\mathbb{R}^n$ are important because they represent the solution set to any homogeneous system of linear equations with n variables. For instance, consider the following system:

$$\begin{cases} 2\,\mathbf{e}_1 - 3\,\mathbf{e}_2 & = 0 \\ \mathbf{e}_1 - 2\,\mathbf{e}_2 + 3\,\mathbf{e}_3 & = 0 \end{cases} \tag{A.1}$$

In the notational convention of this work, $\{\mathbf{e}_i\}_{i=1}^{3}$ is a set of basis vectors defining $\mathbb{R}^3$. Each equation of the system is associated with a plane that passes through the origin of $\mathbb{R}^3$. As depicted in Fig. A.1, the vectors $\mathbf{f}_1 = 2\,\mathbf{e}_1 - 3\,\mathbf{e}_2$ and $\mathbf{f}_2 = \mathbf{e}_1 - 2\,\mathbf{e}_2 + 3\,\mathbf{e}_3$ are the normal vectors (*i.e.,* the dual representation) of such planes. The solution set is the intersection of the planes, which can be computed using the **outer product** ($\wedge$):

$$\left(\mathbf{f}_1 \wedge \mathbf{f}_2\right)^{-*} = 9\,\mathbf{e}_1 + 6\,\mathbf{e}_2 + \mathbf{e}_3. \tag{A.2}$$

Here, the 2-blade spanned as the outer product of $\mathbf{f}_1$ and $\mathbf{f}_2$ is the dual of the solution. The final solution is obtained by taking its undual. The **undual** operation is defined as

$$\mathbf{D}_{\langle n-k \rangle}^{-*} = \mathbf{D}_{\langle n-k \rangle} \rfloor \mathbf{I}_{\langle n \rangle},$$

where $\mathbf{I}_{\langle n \rangle} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n$ is the **unit pseudoscalar** of the n-dimensional space, and $\rfloor$ denotes the **left contraction**. In this example, $n = 3$ and $k = 2$. The resulting subspace will be *zero* when the system has no solution. Note that the technique presented in (A.2) can also be used to solve underdetermined systems (*i.e.* systems with more variables than the number of homogeneous linear equations plus one). In such a case, the result is a subspace whose dimensionality is higher than one.

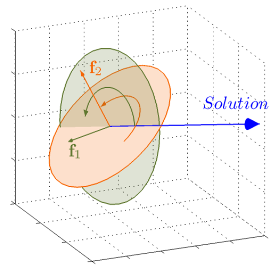


Fig. A.1. The solution of a homogeneous system of linear equations under the Euclidean MOG. Here, $\mathbf{f}_1$ and $\mathbf{f}_2$ are the normal vectors of the planes related to the equations of the system presented in (A.1). The solution set is the vector defined by the intersection of the planes. It is computed by (A.2). The curved arrows indicate the orientation of the planes.

A.3 The Homogeneous Model

The homogeneous (or projective) MOG [4] is similar to the use of homogeneous coordinates in linear algebra. It assumes Euclidean metric and a representational space $\mathbb{R}^{d+1}$ with basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_d\}$. In this MOG, the d-dimensional base space is embedded in $\mathbb{R}^{d+1}$ in such a way that the extra basis vector $\mathbf{e}_0$ is interpreted as the origin of the base space. In Fig. A.2a, the plane parallel to $\mathbf{e}_1 \wedge \mathbf{e}_2$ is the homogeneous representation of the 2-dimensional base space in Fig. A.2b.

In the homogeneous MOG, vectors are geometrically interpreted as points. A **proper point** is a vector defining a finite location $(\alpha_1, \alpha_2, \cdots, \alpha_d)$ in the base space. Such a location is given by the intersection of the 1-blade with the base space (see $\mathbf{e}_0$, $\mathbf{a}$, and $\mathbf{b}$ in Fig. A.2). Unit proper points are written in the form:

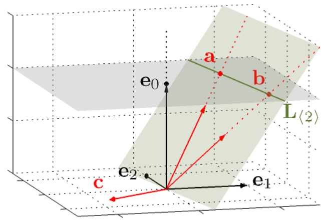
$$\mathbf{p} = \mathbf{e}_0 + \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_d \mathbf{e}_d. \tag{A.3}$$

Note that the coefficient assigned to $\mathbf{e}_0$ in (A.3) is equal to *one*. A general proper point $\gamma\,\mathbf{p}$ is a weighted version of a unit point, and it is interpreted as having the same location (*i.e.,* the weight does not change the geometric interpretation of a blade).

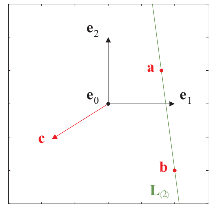
When a vector is parallel to the base space (as $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{c}$ in Fig. A.2) it is called an **improper point**. Such a points can be seen as directions, because they are in the purely directional space $\mathbb{R}^d$ of the representational space $\mathbb{R}^{d+1}$. Unlike proper points, directions have the coefficient of $\mathbf{e}_0$ equal to *zero*:

$$\mathbf{d} = \beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \cdots + \beta_d \mathbf{e}_d. \tag{A.4}$$

Higher dimensional oriented flat subspaces, like straight lines and planes, are spanned as the outer product of proper and improper points. For instance, the line in Fig. A.2 is defined as $\mathbf{L}_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{b}$. Optionally, one can create a



(a) Representational space $\mathbb{R}^{2+1}$



(b) Base space $\mathbb{R}^2$

Fig. A.2. Geometric interpretation of blades in the homogeneous MOG. In (a), the plane parallel to $\mathbf{e}_1 \wedge \mathbf{e}_2$ is the homogeneous representation of the 2-dimensional base space in (b). The geometric interpretation of blades is given by their intersection with the base space. For instance, vectors $\mathbf{e}_0$, $\mathbf{a}$, and $\mathbf{b}$ in (a) are interpreted as proper points in (b), while vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{c}$ in (a) are interpreted as improper points, or directions, in (b). The straight line in (b) is defined by the intersection of a 2-blade with the base space in (a). In such a case, $\mathbf{L}_{\langle 2 \rangle} = \mathbf{a} \wedge \mathbf{b}$.

line from a proper point and a direction by using the same construction. The result will be a 2-blade to be used as a computing primitive.

In the homogeneous MOG, 3-blades are geometrically interpreted as planes. As one would expect, they are defined in d-dimensional base spaces (for $d \geq 3$) as the outer product of: (i) three proper points; (ii) two proper points and one direction; or (iii) one proper point and two directions; as far the vectors are linearly independent. As one can see, the definition of k-flats (for $0 \leq k < d$) in the homogeneous MOG is straightforward. It is based on the outer product of $(k+1)$ vectors. Blades spanned exclusively from improper points are geometrically interpreted as k-flats at infinity. Table A.1 (column H) summarizes the geometric primitives that can be represented as blades in the homogeneous MOG when a 2-dimensional ($d = 2$) or a 3-dimensional ($d = 3$) base space is assumed.

A.4 The Conformal Model

In the conformal MOG [5,6] blades can be geometrically interpreted not only as directions and flats, like in homogeneous MOG, but also as **rounds** (*e.g.*, point pairs, circles, spheres, and their higher-dimensional counterparts) and **tangent subspaces**.

The representational vector space $\mathbb{R}^{d+2}$ of the conformal MOG is defined from the basis vectors $\{\mathbf{o}, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_d, \boldsymbol{\infty}\}$, where the d-dimensional base space is enhanced with two extra dimensions: $\mathbf{o}$, a null vector interpreted as the origin point; and $\boldsymbol{\infty}$, a null vector interpreted as the point at infinity. They are null vectors due to the special metric assumed in the conformal MOG. The multiplication table for the vector inner product of the basis vectors is:

$\cdot$	$\mathbf{o}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\cdots$	$\mathbf{e}_d$	$\boldsymbol{\infty}$
$\mathbf{o}$	0	0	0	$\cdots$	0	-1
$\mathbf{e}_1$	0	1	0	$\cdots$	0	0
$\mathbf{e}_2$	0	0	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\mathbf{e}_d$	0	0	0	$\cdots$	1	0
$\boldsymbol{\infty}$	-1	0	0	$\cdots$	0	0

Note that $\mathbf{o} \cdot \mathbf{o} = \boldsymbol{\infty} \cdot \boldsymbol{\infty} = 0$, while $\mathbf{o} \cdot \boldsymbol{\infty} = -1$. One consequence of this definition is that the inner product of two unit finite points (*i.e.,* points at a finite distance from the origin) is given in terms of the square of the Euclidean distance between them:

$$\mathbf{p} \cdot \mathbf{q} = -\frac{1}{2} \sum_{i=1}^{d} (\alpha_i - \beta_i)^2, \tag{A.5}$$

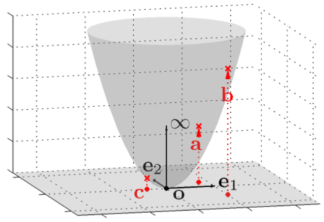
where $(\alpha_1, \alpha_2, \cdots, \alpha_d)$ and $(\beta_1, \beta_2, \cdots, \beta_d)$ are the location of points $\mathbf{p}$ and $\mathbf{q}$, respectively. This way, finite points are also null vectors (*i.e.,* $\mathbf{p} \cdot \mathbf{p} = 0$).

Unit finite points are written in the form:

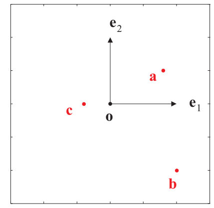
$$\mathbf{p} = \mathbf{o} + \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_d \mathbf{e}_d + \frac{1}{2} \sum_{i=1}^{d} \left(\alpha_i^2\right) \boldsymbol{\infty}, \tag{A.6}$$

while general finite points are weighted points $(\gamma \mathbf{p})$ having the same location. Fig. A.3a shows that the set of all unit finite points in a 2-dimensional base space defines a paraboloid in the $\boldsymbol{\infty}$-direction of the representational space. In this example, the representational 4-dimensional vector space is presented as a 3-dimensional homogeneous space, where $\mathbf{o}$ is treated as the homogeneous coordinate. The base space shown in Fig. A.3b corresponds to the plane at the bottom of Fig. A.3a. Note that the paraboloid touches the base space at its origin, and that the location of finite points (*e.g.,* $\mathbf{o}$, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$) is given by their orthogonal projection onto the base space.

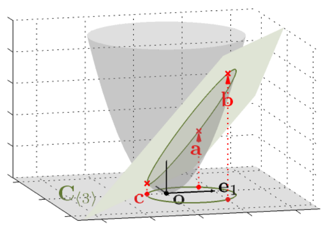
From the outer product of two, three, and four finite points one builds, respectively, point pairs, circles, and spheres. So, the construction of k-spheres (for $0 \leq k < d$) is straightforward. It is achieved from the outer product of $(k+2)$ points. Fig. A.3d shows a circle defined as $\mathbf{C}_{\langle 3 \rangle} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$. Note in Fig. A.3c that such circle is a cross section of the paraboloid, orthogonally projected onto the base space. Refer to [3, Appendix B] for formulas defining k-spheres from their usual center-radius parameterization.



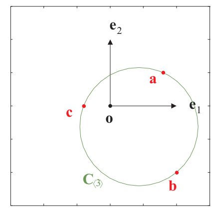
(a) Representational space $\mathbb{R}^{2+2}$



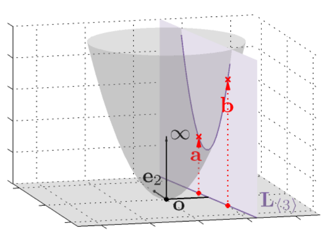
(b) Base space $\mathbb{R}^2$



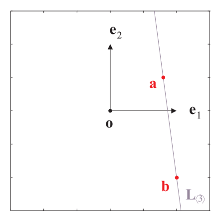
(c) Representational space $\mathbb{R}^{2+2}$



(d) Base space $\mathbb{R}^2$



(e) Representational space $\mathbb{R}^{2+2}$



(f) Base space $\mathbb{R}^2$

Fig. A.3. Geometric interpretation of blades in the conformal MOG. The representation of a 2-dimensional base space is shown on the left. In such a representation, the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\boldsymbol{\infty}$ are seen as a homogeneous space having the basis vector $\mathbf{o}$ as homogeneous coordinate. The 2-dimensional base space (on the right) corresponds to the plane at the bottom of the images on the left. Points on the paraboloid are interpreted as finite points in the base space (see $\mathbf{o}$, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in (a) and (b)). As depicted in (c) and (d), the circle defined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is computed as $\mathbf{C}_{\langle 3 \rangle} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$. In (e) and (f), the straight line that passes thought $\mathbf{a}$ and $\mathbf{b}$ is defined as $\mathbf{L}_{\langle 3 \rangle} = \mathbf{a} \wedge \mathbf{b} \wedge \boldsymbol{\infty}$.

In the conformal MOG the base space is "closed". It means that $\boldsymbol{\infty}$ is the unique point at infinity, with a well defined location that one can approach from *any* direction. So, $\boldsymbol{\infty}$ is common to all flat subspaces, because they stretch to infinity. This way, straight lines and planes are built as the outer product of $\boldsymbol{\infty}$ with, respectively, two and three finite points. For instance, the line passing through $\mathbf{a}$ to $\mathbf{b}$ in Fig. A.3e is computed as $\mathbf{L}_{\langle 3 \rangle} = \mathbf{a} \wedge \mathbf{b} \wedge \boldsymbol{\infty}$. It is important to note that all the equations for construction of flat subspaces in the conformal MOG are backward compatible with the ones from the homogeneous MOG, but including $\boldsymbol{\infty}$.

In Fig. A.3e, the cross section defined by $\mathbf{L}_{\langle 3\rangle}$ is a parabola whose orthogonal projection on the base space is a straight line (Fig. A.3f). Now, note how similar $\mathbf{C}_{\langle 3\rangle}$ and $\mathbf{L}_{\langle 3\rangle}$ are (Figs. A.3c and A.3e). Both are 3-blades which define cross sections in the paraboloid. In fact, the projection of both cross sections can be interpreted as circles in the base space, where $\mathbf{L}_{\langle 3\rangle}$ (Fig. A.3f) is a circle with infinite radius. Such a generality on the geometric interpretation of blades is explored by our detection framework [7] in order to perform concurrent detection of subspaces with different interpretations but with the same dimensionality.

In order to be interpreted as a direction, a blade must have only directional properties and no locational aspects. The location of a blade is defined in terms of the assumed origin point $\mathbf{o}$. Therefore, directions (also called **free blades**) are built as $\mathbf{A}_{\langle k\rangle} \wedge \boldsymbol{\infty}$, where $\mathbf{A}_{\langle k\rangle} \subset (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_d)$. This is the natural extension of blades, which are interpreted as directions in the homogeneous MOG, to the conformal MOG.

The final type of conformal blade is the **tangent subspace**. As the name suggests, such primitives are tangent to something. In such a case, they encode the subspace tangent to rounds or flats at a given location. Therefore, tangent subspaces have a point-like interpretation, and also direction information assigned to them. For a given round (or flat) $\mathbf{X}_{\langle k\rangle}$ passing thought the point $\mathbf{p}$, the tangent subspace at the location of $\mathbf{p}$ is $\mathbf{T}_{\langle k-1\rangle} = \mathbf{p} \rfloor \widehat{\mathbf{X}}_{\langle k\rangle}$, for $\widehat{\mathbf{X}}_{\langle k\rangle} = (-1)^k \mathbf{X}_{\langle k\rangle}$ denoting the **grade involution** of $\mathbf{X}_{\langle k\rangle}$. The general equation for tangents subspaces is:

$$\mathbf{T}_{\langle k+1\rangle} = \mathbf{p} \wedge (-\mathbf{p} \rfloor (\widehat{\mathbf{A}}_{\langle k\rangle}\, \boldsymbol{\infty})),$$

where $\mathbf{A}_{\langle k\rangle} \subset (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_d)$ defines the direction.

Table A.1 (column C) summarizes the geometric primitives that can be represented as blades in the conformal MOG when a 2-dimensional ($d = 2$) or a 3-dimensional ($d = 3$) base space is assumed.

A.5 The Conic Model

Perwass and Forstner [8] show how to embed a 2-dimensional base space $\mathbb{R}^2$ in a 6-dimensional representation space $\mathbb{R}^6$ in order to encode **conic sections** (*e.g.,* circle, ellipse, straight line, hyperbola, parallel line pair, and intersecting line pair) as blades.

This MOG assumes Euclidean metric and the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}$ for $\mathbb{R}^6$.

Unit finite points are written in the form:

$$\mathbf{p} = \alpha\,\mathbf{e}_1 + \beta\,\mathbf{e}_2 + \frac{1}{\sqrt{2}}\,\mathbf{e}_3 + \frac{1}{\sqrt{2}}\,\alpha^2\,\mathbf{e}_4 + \frac{1}{\sqrt{2}}\,\beta^2\,\mathbf{e}_5 + \alpha\,\beta\,\mathbf{e}_6,$$

where α and β define the location of a point in the 2-dimensional base space.

From the outer product of two, three, and four finite points one builds point pairs, point triplets, and point quadruplets. The outer product of five distinct points defines a 5-blade interpreted as one of the types of conic sections. The geometric primitives encoded as k-blades in such a MOG for $d = 2$ are summarized in Table A.1 (column N).

References

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[8] C. Perwass, W. Förstner, Geometric properties for incomplete data, Springer-Verlag, 2006, Ch. Uncertain geometry with circles, spheres and conics, pp. 23–41.

Table A.1
Geometric interpretation of subspaces according to four MOGs (E: Euclidean, H: Homogeneous, C: Conformal, and N: Conic). n indicates the dimensionality of the representational space, and d is the dimensionality of the base space. The extra dimensions (added to d) are imposed by the MOG. A 2-dimensional base space is assumed for this table. The only exceptions are for shapes marked with †. In such cases, the base space is 3-dimensional. The rows group shapes according to their class (*i.e.,* frees, flats, rounds, tangents, point sets, and conic sections). The table entries show the dimensionality of the subspaces. This table can be extended to higher-dimensions, to other geometric shapes, and to other MOGs.

		E	**H**	**C**	**N**
	$n \rightarrow$	d	$d+1$	$d+2$	6
Frees	1-D direction	1	1	2	
	2-D direction	2	2	3	
	3-D direction†	3	3	4	
Flats	Flat point		1	2	
	Straight line		2	3	
	Plane†		3	4	
Rounds	Point pair			2	
	Circle			3	
	Sphere†			4	
Tangents	Point			1	
	1-D tangent direction			2	
	2-D tangent direction†			3	
Point Sets	Point				1
	Point pair				2
	Point triplet				3
	Point quadruplet				4
Conics	Circle, Ellipse, Straight line, Hyperbola, Parallel line pair, Intersecting line pair				5